

# FRAGMENTED DEFORMATIONS OF PRIMITIVE MULTIPLE CURVES

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RESUME. A *primitive multiple curve* is a Cohen-Macaulay irreducible projective curve  $Y$  that can be locally embedded in a smooth surface, and such that  $Y_{red}$  is smooth.

The subject of this paper is the study of deformations of  $Y$  in curves with smooth irreducible components, when the number of components is maximal (it is then the multiplicity  $n$  of  $Y$ ).

We are particularly interested in deformations in  $n$  disjoint smooth irreducible components, which are called *fragmented deformations*. We describe them completely. We give also a characterization of primitive multiple curves having a fragmented deformation.

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## 1. INTRODUCTION

A *primitive multiple curve* is an algebraic variety  $Y$  over  $\mathbb{C}$  which is Cohen-Macaulay, such that the induced reduced variety  $C = Y_{red}$  is a smooth projective irreducible curve, and that every closed point of  $Y$  has a neighbourhood that can be embedded in a smooth surface. These curves have been defined and studied by C. Bănică and O. Forster in [1]. The simplest examples are infinitesimal neighbourhoods of projective smooth curves embedded in a smooth surface (but most primitive multiple curves cannot be globally embedded in smooth surfaces, cf. [2], theorem 7.1).

Let  $Y$  be a primitive multiple curve with associated reduced curve  $C$ , and suppose that  $Y \neq C$ . Let  $\mathcal{I}_C$  be the ideal sheaf of  $C$  in  $Y$ . The *multiplicity* of  $Y$  is the smallest integer  $n$  such that  $\mathcal{I}_C^n = 0$ . We have then a filtration

$$C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$$

where  $C_i$  is the subscheme corresponding to the ideal sheaf  $\mathcal{I}_C^i$  and is a primitive multiple curve of multiplicity  $i$ . The sheaf  $\mathcal{L} = \mathcal{I}_C/\mathcal{I}_C^2$  is a line bundle on  $C$ , called the *line bundle on  $C$  associated to  $Y$* .

**1.1. History and motivation** – The deformations of double (i.e. of multiplicity 2) primitive multiple curves (also called *ribbons*) in smooth projective curves have been studied in [14]. In this paper we are interested in deformations of primitive multiple curves  $Y$  of any multiplicity  $n \geq 2$  in reduced curves having exactly  $n$  components which are smooth ( $n$  is the maximal number of components of deformations of  $Y$ ). In this case the number of intersection points of two components is exactly  $-\deg(L)$ . We give some results in the general case (no assumption on  $\deg(L)$ ) and treat more precisely the case  $\deg(L) = 0$ , i.e. deformations of  $Y$  in curves having exactly  $n$  disjoint irreducible components.

Let  $\pi : \mathcal{C} \rightarrow S$  be a flat projective morphism of algebraic varieties,  $P$  a closed point of  $S$  such that  $\pi^{-1}(P) \simeq Y$ ,  $\mathcal{O}_{\mathcal{C}}(1)$  a very ample line bundle on  $\mathcal{C}$  and  $P$  a polynomial in one variable with rational coefficients. Let

$$\tau : \mathcal{M}_{\mathcal{O}_{\mathcal{C}}(1)}(P) \longrightarrow S$$

be the corresponding relative moduli space of semi-stable sheaves (parametrizing the semi-stable sheaves on the fibers of  $\pi$  with Hilbert polynomial  $P$  with respect to the restriction of  $\mathcal{O}_{\mathcal{C}}(1)$ , cf. [20]). In general  $\tau$  is not flat (some other examples on non flat relative moduli spaces are given in [17]). For example, if the family  $\mathcal{C}$  contains smooth fibers, it is impossible to deform the stable sheaf  $\mathcal{O}_C$  on  $Y$  in sheaves on the smooth fibers. I conjecture that  $\tau$  is flat if all the fibers of  $\tau$  are reduced with exactly  $n$  components. The reason is that the generic structure of torsion free sheaves on  $Y$  (cf. [8]) is more complicated than on smooth curves, and is somehow similar to the generic structure of torsion free sheaves on reducible reduced curves (cf. [23], [24]).

**1.2. Maximal reducible deformations** – Let  $(S, P)$  be a germ of smooth curve. Let  $Y$  be a primitive multiple curve of multiplicity  $n \geq 2$  and  $k > 0$  an integer. Let  $\pi : \mathcal{C} \rightarrow S$  be a flat morphism, where  $\mathcal{C}$  is a reduced algebraic variety, such that

- For every closed point  $s \in S$  such that  $s \neq P$ , the fiber  $\mathcal{C}_s$  has  $k$  irreducible components, which are smooth and transverse, and any three of these components have no common point.
- The fiber  $\mathcal{C}_P$  is isomorphic to  $Y$ .

We show that by making a change of variable, i.e. by considering a suitable germ  $(S', P')$  and a non constant morphism  $\tau : S' \rightarrow S$ , and replacing  $\pi$  with  $\pi^*\mathcal{C} \rightarrow S'$ , we can suppose that  $\mathcal{C}$  has exactly  $k$  irreducible components, inducing on every fiber  $\mathcal{C}_s$ ,  $s \neq P$  the  $k$  irreducible components of  $\mathcal{C}_s$ . In this case  $\pi$  is called a *reducible deformation of  $Y$  of length  $k$* .

We show that  $k \leq n$ . We say that  $\pi$  (or  $\mathcal{C}$ ) is a *maximal reducible deformation of  $Y$*  if  $k = n$ .

Suppose that  $\pi$  is a maximal reducible deformation of  $Y$ . We show that if  $\mathcal{C}'$  is the union of  $i > 0$  irreducible components of  $\mathcal{C}$ , and  $\pi' : \mathcal{C}' \rightarrow S$  is the restriction of  $\pi$ , then  $\pi'^{-1}(P) \simeq C_i$ , and  $\pi'$  is a maximal reducible deformation of  $C_i$ . Let  $s \in S \setminus \{P\}$ . We prove that the irreducible components of  $\mathcal{C}_s$  have the same genus as  $C$ . Moreover, if  $D_1, D_2$  are distinct irreducible components of  $\mathcal{C}_s$ , then  $D_1 \cap D_2$  consists of  $-\deg(L)$  points.

**1.3. Fragmented deformations (definition)** – Let  $Y$  be a primitive multiple curve of multiplicity  $n \geq 2$  and  $\pi : \mathcal{C} \rightarrow S$  a maximal reducible deformation of  $Y$ . We call it a *fragmented deformation* of  $Y$  if  $\deg(L) = 0$ , i.e. if for every  $s \in S \setminus \{P\}$ ,  $\mathcal{C}_s$  is the disjoint union of  $n$  smooth curves. In this case  $\mathcal{C}$  has  $n$  irreducible components  $\mathcal{C}_1, \dots, \mathcal{C}_n$  which are smooth surfaces.

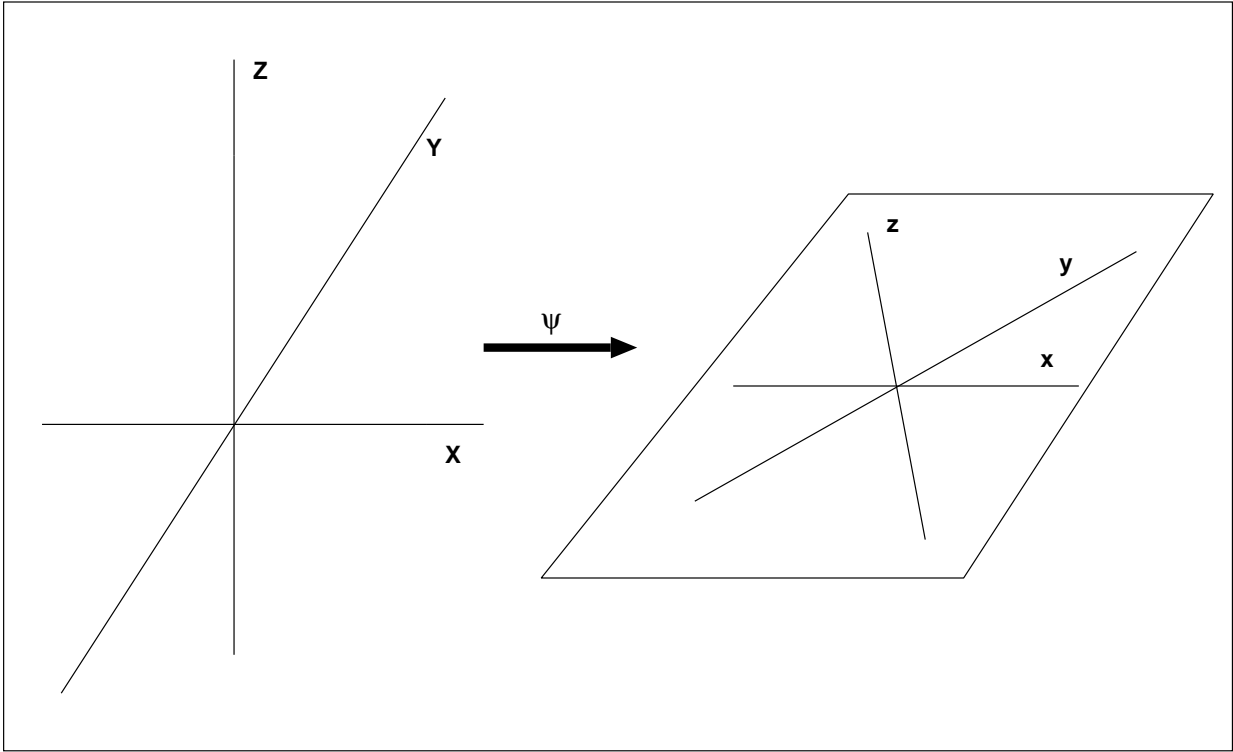
The variety  $\mathcal{C}$  appears as a particular case of a *glueing* of  $\mathcal{C}_1, \dots, \mathcal{C}_n$  along  $C$  (cf. 4.1.5). We prove (proposition 4.1.5) that such a glueing  $\mathcal{D}$  is a fragmented deformation of a primitive multiple curve if and only if every closed point in  $C$  has a neighbourhood in  $\mathcal{D}$  that can be embedded in a smooth variety of dimension 3. The simplest glueing is the trivial or *initial glueing*  $\mathcal{A}$ . An open subset  $U$  of  $\mathcal{A}$  (and  $\mathcal{C}$ ) is given by open subsets  $U_1, \dots, U_n$  of  $\mathcal{C}_1, \dots, \mathcal{C}_n$  respectively, having the same intersection with  $C$ , and

$$\mathcal{O}_{\mathcal{A}}(U) = \{(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C}_1}(U \cap \mathcal{C}_1) \times \dots \times \mathcal{O}_{\mathcal{C}_n}(U \cap \mathcal{C}_n); \alpha_1|_C = \dots = \alpha_n|_C\},$$

and  $\mathcal{O}_{\mathcal{C}}(U)$  appears as a subalgebra of  $\mathcal{O}_{\mathcal{A}}(U)$ , hence we have a canonical morphism  $\mathcal{A} \rightarrow \mathcal{C}$ .

We can view elements of  $\mathcal{O}_{\mathcal{C}}(U)$  as  $n$ -tuples  $(\alpha_1, \dots, \alpha_n)$ , with  $\alpha_i \in \mathcal{O}_{\mathcal{C}_i}(U \cap \mathcal{C}_i)$ . In particular we can write  $\pi = (\pi_1, \dots, \pi_n)$ .

**1.4. A simple analogy** – Consider  $n$  copies of  $\mathbb{C}$  glued at 0. Two extreme examples appear : the trivial glueing  $\mathcal{A}_0$  (the set of coordinate lines in  $\mathbb{C}^n$ ), and a set  $\mathcal{C}_0$  of  $n$  lines in  $\mathbb{C}^2$ . We can easily construct a bijective morphism  $\Psi : \mathcal{A}_0 \rightarrow \mathcal{C}_0$  sending each coordinate line to a line in the plane



But the two schemes are of course not isomorphic : the maximal ideal of 0 in  $\mathcal{A}_0$  needs  $n$  generators, but 2 are enough for the maximal ideal of 0 in  $\mathcal{C}_0$ .

Let  $\pi_{\mathcal{C}_0} : \mathcal{C}_0 \rightarrow \mathbb{C}$  be a morphism sending each component linearly onto  $\mathbb{C}$ , and  $\pi_{\mathcal{A}_0} = \pi_{\mathcal{C}_0} \circ \Psi : \mathcal{A}_0 \rightarrow \mathbb{C}$ . The difference of  $\mathcal{A}_0$  and  $\mathcal{C}_0$  can be also seen by using the fibers of 0 : we have

$$\pi_{\mathcal{C}_0}^{-1}(0) \simeq \operatorname{spec}(\mathbb{C}[t]/(t^n)) \quad \text{and} \quad \pi_{\mathcal{A}_0}^{-1}(0) \simeq \operatorname{spec}(\mathbb{C}[t_1, \dots, t_n]/(t_1, \dots, t_n)^2).$$

Let  $\mathcal{D}$  a general glueing of  $n$  copies of  $\mathbb{C}$  at 0, such that there exists a morphism  $\pi : \mathcal{D} \rightarrow \mathbb{C}$  inducing the identity on each copy of  $\mathbb{C}$ . It is easy to see that we have  $\pi^{-1}(0) \simeq \operatorname{spec}(\mathbb{C}[t]/(t^n))$  if and only if some neighbourhood of 0 in  $\mathcal{D}$  can be embedded in a smooth surface.

**1.5. Fragmented deformations (main properties)** – Let  $\pi : \mathcal{C} \rightarrow S$  be a fragmented deformation of  $Y = C_n$ . Let  $I \subset \{1, \dots, n\}$  be a proper subset,  $I^c$  its complement, and  $\mathcal{C}_I \subset \mathcal{C}$  the subscheme union of the  $\mathcal{C}_i, i \in I$ . We prove (theorem 4.3.7) that the ideal sheaf  $\mathcal{I}_{\mathcal{C}_I}$  of  $\mathcal{C}_I$  is isomorphic to  $\mathcal{O}_{\mathcal{C}_{I^c}}$ .

In particular, the ideal sheaf  $\mathcal{I}_{\mathcal{C}_i}$  of  $\mathcal{C}_i$  is generated by a single regular function on  $\mathcal{C}$ . We show that we can find such a generator such that for  $1 \leq j \leq n, j \neq i$ , its  $j$ -th coordinate can be written as  $\alpha \pi_j^p$ , with  $p > 0$  and  $\alpha \in H^0(\mathcal{O}_S)$  such that  $\alpha(P) \neq 0$ . We can then suppose that  $\alpha = 1$ , and the generator can be written as

$$\mathbf{u}_{ij} = (u_1, \dots, u_m),$$

with

$$u_i = 0, \quad u_m = \alpha_{ij}^{(m)} \pi_m^{p_{im}} \text{ for } m \neq i, \quad \alpha_{ij}^{(j)} = 1.$$

The constants  $\mathbf{a}_{ij}^{(m)} = \alpha_{ij|C}^{(m)} \in \mathbb{C}$  have interesting properties (propositions 4.5.2, 4.4.6). Let  $p_{ii} = 0$  for  $1 \leq i \leq n$ . The symmetric matrix  $(p_{ij})_{1 \leq i, j \leq n}$  is called the *spectrum* of  $\pi$  (or  $\mathcal{C}$ ).

It follows also from the fact that  $\mathcal{I}_{\mathcal{C}_i} = (\mathbf{u}_{ij})$  that  $Y$  is a *simple* primitive multiple curve, i.e. the ideal sheaf of  $C$  in  $Y = C_n$  is isomorphic to  $\mathcal{O}_{C_{n-1}}$ . Conversely, we show in theorem 4.7.1 that if  $Y$  is a simple primitive multiple curve, then there exists a fragmented deformation of  $Y$ .

We give in 4.4 and 4.5 a way to construct fragmented deformations by induction on  $n$ . This is used later to prove statements on fragmented deformations by induction on  $n$ .

**1.6.  $n$ -stars and structure of fragmented deformations** – A  $n$ -star of  $(S, P)$  is a glueing  $\mathcal{S}$  of  $n$  copies of  $S$  at  $P$ , together with a morphism  $\pi : \mathcal{S} \rightarrow S$  which is an identity on each copy of  $S$ . All the  $n$ -stars have the same underlying Zariski topological space  $S(n)$ .

A  $n$ -star is called *oblate* if some neighbourhood of  $P$  can be embedded in a smooth surface. This is the case if and only if  $\pi^{-1}(0) \simeq \operatorname{spec}(\mathbb{C}[t]/(t^n))$ .

Oblate  $n$ -stars are analogous to fragmented deformations and simpler. We provide a way to build oblate  $n$ -stars by induction on  $n$ .

Let  $\pi : \mathcal{C} \rightarrow S$  be a fragmented deformation of  $Y = C_n$ . We associate to it an oblate  $n$ -star  $\mathcal{S}$  of  $S$  : for every open subset  $U$  of  $S(n)$ ,  $\mathcal{O}_{\mathcal{S}}(U)$  is the set of  $(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C}}(U)$  such that  $\alpha_i \in \mathcal{O}_S(\pi_i(U \cap \mathcal{C}_i))$  for  $1 \leq i \leq n$ . We obtain also a canonical morphism

$$\Pi : \mathcal{C} \longrightarrow \mathcal{S}.$$

We prove (theorem 5.6.2) that  $\Pi$  is flat. Hence it is a flat family of smooth curves, with  $\Pi^{-1}(P) = C$ . The converse is also true, i.e. starting from an oblate  $n$ -star of  $S$  and a flat

family of smooth curves parametrized by it, we obtain a fragmented deformation of a multiple primitive curve of multiplicity  $n$ .

**1.7. Fragmented deformations of double curves** – Let  $Y = C_2$  be a primitive double curve,  $C$  its associated smooth curve,  $\pi : \mathcal{C} \rightarrow S$  a fragmented deformation of  $Y$ , of spectrum  $\begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$ , and  $\mathcal{C}_1, \mathcal{C}_2$  the irreducible components of  $\mathcal{C}$ . For  $i = 1, 2$ ,  $q > 0$ , let  $C_i^q$  be the infinitesimal neighbourhood of order  $q$  of  $C$  in  $\mathcal{C}_i$  (defined by the ideal sheaf  $(\pi_i^q)$ ). It is a primitive multiple curve of multiplicity  $q$ .

It follows from 4.3.5 that  $C_1^p$  and  $C_2^p$  are isomorphic, and  $C_1^{p+1}, C_2^{p+1}$  are two extensions of  $C_1^p$  in primitive multiple curves of multiplicity  $p + 1$ . According to [6] these extensions are parametrized by an affine space with associated vector space  $H^1(C, T_C)$  (where  $T_C$  is the tangent bundle of  $C$ ). Let  $w \in H^1(C, T_C)$  be the vector from  $C_1^{p+1}$  to  $C_2^{p+1}$ .

Similarly, the primitive double curves with associated smooth curve  $C$  such that  $\mathcal{I}_C \simeq \mathcal{O}_C$  are parametrized by  $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$  (cf. [2], [6]).

We prove in theorem 6.0.5 that the point of  $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$  corresponding to  $C_2$  is  $\mathbb{C}w$ .

**1.8. Notation:** Let  $X$  be an algebraic variety and  $Y \subset X$  a closed subvariety. We will denote by  $\mathcal{I}_{Y,X}$  (or  $\mathcal{I}_Y$  if there is no risk of confusion) the ideal sheaf of  $Y$  in  $X$ .

## 2. PRELIMINARIES

### 2.1. LOCAL EMBEDDINGS IN SMOOTH VARIETIES

**2.1.1. Proposition:** *Let  $X$  be an algebraic variety,  $x$  a closed point of  $X$  and  $n$  a positive integer. Then the two following properties are equivalent:*

- (i) *There exists a neighbourhood  $U$  of  $x$  and an embedding  $U \subset Z$  in a smooth variety of dimension  $n$ .*
- (ii) *The  $\mathcal{O}_{X,x}$ -module  $m_{X,x}$  (maximal ideal of  $x$ ) can be generated by  $n$  elements.*
- (iii) *We have  $\dim_{\mathbb{C}}(m_{X,x}/m_{X,x}^2) \leq n$ .*

*Proof.* It is obvious that (i) implies (ii), and (ii),(iii) are equivalent according to Nakayama's lemma. It remains to prove that (iii) implies (i).

Suppose that (iii) is true. There exists an integer  $N$  and an embedding  $X \subset \mathbb{P}_N$ . Let  $\mathcal{I}_X$  be the ideal sheaf of  $X$  in  $\mathbb{P}_N$ . Let  $p$  be the biggest integer such that there exists  $f_1, \dots, f_p \in \mathcal{I}_{X,x}$  whose images in the  $\mathbb{C}$ -vector space  $m_{\mathbb{P}_N,x}/m_{\mathbb{P}_N,x}^2$  are linearly independant. Then we have

$$\mathcal{I}_{X,x} \subset (f_1, \dots, f_p) + m_{\mathbb{P}_N,x}^2.$$

In fact, let  $f \in \mathcal{I}_{X,x}$ . Since  $p$  is maximal, the image of  $f$  in  $m_{\mathbb{P}_N,x}/m_{\mathbb{P}_N,x}^2$  is a linear combination of those of  $f_1, \dots, f_n$ . Hence we can write

$$f = \sum_{i=1}^p \lambda_i f_i + g, \quad \text{with } \lambda_i \in \mathbb{C}, g \in m_{\mathbb{P}_N,x}^2,$$

and our assertion is proved. It follows that we have a surjective morphism

$$\alpha : \mathcal{O}_{X,x}/m_{X,x}^2 \longrightarrow \mathcal{O}_{\mathbb{P}_N,x}/((f_1, \dots, f_p) + m_{\mathbb{P}_N,x}^2).$$

We have

$$\dim_{\mathbb{C}}(\mathcal{O}_{X,x}/m_{X,x}^2) \leq n+1, \quad \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{P}_N,x}/((f_1, \dots, f_p) + m_{\mathbb{P}_N,x}^2)) = N-p+1.$$

Hence  $N-p+1 \leq n+1$ , i.e.  $p \geq N-n$ . We can take for  $Z$  a neighbourhood of  $x$  in the subvariety of  $\mathbb{P}_N$  defined by  $f_1, \dots, f_{N-n}$ , which is smooth at  $x$ .  $\square$

## 2.2. FLAT FAMILIES OF COHERENT SHEAVES

Let  $(S, P)$  be a smooth germ of curve and  $t \in \mathcal{O}_{S,P}$  a generator of the maximal ideal. Let  $\pi : X \rightarrow S$  be a flat morphism. If  $\mathcal{E}$  is a coherent sheaf on  $X$ ,  $\mathcal{E}$  is flat on  $S$  at  $x \in \pi^{-1}(P)$  if and only if the multiplication by  $t : \mathcal{E}_x \rightarrow \mathcal{E}_x$  is injective. In particular the multiplication by  $t : \mathcal{O}_x \rightarrow \mathcal{O}_x$  is injective.

**2.2.1. Lemma:** *Let  $\mathcal{E}$  be a coherent sheaf on  $X$  flat on  $S$ . Then, for every open subset  $U$  of  $X$ , the restriction  $\mathcal{E}(U) \rightarrow \mathcal{E}(U \setminus \pi^{-1}(P))$  is injective.*

*Proof.* Let  $s \in \mathcal{E}(U)$  whose restriction to  $U \setminus \pi^{-1}(P)$  vanishes. We must show that  $s = 0$ . By covering  $U$  with smaller open subsets we can suppose that  $U$  is affine:  $U = \text{spec}(A)$ . Hence  $U \setminus \pi^{-1}(P) = \text{spec}(A_t)$ . Let  $M = \mathcal{E}(U)$ , it is an  $A$ -module. We have  $\mathcal{E}|_U = \widehat{M}$  and  $\mathcal{E}(U \setminus \pi^{-1}(P)) = M_t$ . Hence if the restriction of  $s$  to  $U \setminus \pi^{-1}(P)$  vanishes, there exists an integer  $n > 0$  such that  $t^n s = 0$ . Since the multiplication by  $t$  is injective (because  $\mathcal{E}$  is flat on  $S$ ), we have  $s = 0$ .  $\square$

Let  $\mathcal{E}$  be a coherent sheaf on  $X$  flat on  $S$ . Let  $\mathcal{F} \subset \mathcal{E}|_{X \setminus \pi^{-1}(P)}$  be a subsheaf. For every open subset  $U$  of  $X$  we denote by  $\overline{\mathcal{F}}(U)$  the subset of  $\mathcal{F}(U \setminus \pi^{-1}(P))$  of elements that can be extended to sections of  $\mathcal{E}$  on  $U$ . If  $V \subset U$  is an open subset, the restriction  $\mathcal{F}(U \setminus \pi^{-1}(P)) \rightarrow \mathcal{F}(V \setminus \pi^{-1}(P))$  induces a morphism  $\overline{\mathcal{F}}(U) \rightarrow \overline{\mathcal{F}}(V)$ .

**2.2.2. Proposition:**  *$\overline{\mathcal{F}}$  is a subsheaf of  $\mathcal{E}$ , and  $\mathcal{E}/\overline{\mathcal{F}}$  is flat on  $S$ .*

*Proof.* To prove the first assertion, we must show that if  $U$  is an open subset of  $X$  and  $(U_i)_{i \in I}$  is an open cover of  $U$ , then

- (i) If  $s \in \overline{\mathcal{F}}(U)$  is such that for every  $i$  we have  $s|_{U_i} = 0$ , then  $s = 0$ .
- (ii) For every  $i \in I$  let  $s_i \in \overline{\mathcal{F}}(U_i)$ . Then if for all  $i, j$  we have  $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ , then there exists  $s \in \overline{\mathcal{F}}(U)$  such that for every  $i \in I$  we have  $s|_{U_i} = s_i$ .

This follows easily from lemma 2.2.1.

Now we prove that  $\mathcal{E}/\overline{\mathcal{F}}$  is flat on  $S$ . Let  $x \in \pi^{-1}(P)$  and  $u \in (\mathcal{E}/\overline{\mathcal{F}})_x$  such that  $tu = 0$ . We must show that  $u = 0$ . Let  $v \in \mathcal{E}_x$  over  $u$ . Then we have  $tv \in \overline{\mathcal{F}}_x$ . Let  $U$  be a neighbourhood of  $x$  such that  $tv$  comes from  $w \in \overline{\mathcal{F}}(U)$ . This means that  $w|_{U \setminus \pi^{-1}(P)} \in \mathcal{F}(U \setminus \pi^{-1}(P))$ . Since  $t$  is invertible on  $U \setminus \pi^{-1}(P)$  we can write  $w = tw'$ , with  $w' \in \mathcal{F}(U \setminus \pi^{-1}(P))$ . We have then  $w' = v$  on  $U \setminus \pi^{-1}(P)$ . Hence  $v \in \overline{\mathcal{F}}_x$  and  $u = 0$ .  $\square$

### 2.3. PRIMITIVE MULTIPLE CURVES

(cf. [1], [5], [8]).

Let  $C$  be a smooth connected projective curve. A *multiple curve with support  $C$*  is a Cohen-Macaulay scheme  $Y$  such that  $Y_{red} = C$ .

Let  $n$  be the smallest integer such that  $Y = C^{(n-1)}$ ,  $C^{(k-1)}$  being the  $k$ -th infinitesimal neighbourhood of  $C$ , i.e.  $\mathcal{I}_{C^{(k-1)}} = \mathcal{I}_C^k$ . We have a filtration  $C = C_1 \subset C_2 \subset \dots \subset C_n = Y$  where  $C_i$  is the biggest Cohen-Macaulay subscheme contained in  $Y \cap C^{(i-1)}$ . We call  $n$  the *multiplicity* of  $Y$ .

We say that  $Y$  is *primitive* if, for every closed point  $x$  of  $C$ , there exists a smooth surface  $S$ , containing a neighbourhood of  $x$  in  $Y$  as a locally closed subvariety. In this case,  $L = \mathcal{I}_C/\mathcal{I}_{C_2}$  is a line bundle on  $C$  and we have  $\mathcal{I}_{C_j} = \mathcal{I}_X^j$ ,  $\mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}} = L^j$  for  $1 \leq j < n$ . We call  $L$  the line bundle on  $C$  *associated* to  $Y$ . Let  $P \in C$ . Then there exists elements  $y, t$  of  $m_{S,P}$  (the maximal ideal of  $\mathcal{O}_{S,P}$ ) whose images in  $m_{S,P}/m_{S,P}^2$  form a basis, and such that for  $1 \leq i < n$  we have  $\mathcal{I}_{C_i,P} = (y^i)$ .

The simplest case is when  $Y$  is contained in a smooth surface  $S$ . Suppose that  $Y$  has multiplicity  $n$ . Let  $P \in C$  and  $f \in \mathcal{O}_{S,P}$  a local equation of  $C$ . Then we have  $\mathcal{I}_{C_i,P} = (f^i)$  for  $1 < j \leq n$ , in particular  $\mathcal{I}_{Y,P} = (f^n)$ , and  $L = \mathcal{O}_C(-C)$ .

We will note  $\mathcal{O}_n = \mathcal{O}_{C_n}$  and we will see  $\mathcal{O}_i$  as a coherent sheaf on  $C_n$  with schematic support  $C_i$  if  $1 \leq i < n$ .

If  $\mathcal{E}$  is a coherent sheaf on  $Y$  one defines its *generalized rank*  $R(\mathcal{E})$  and *generalized degree*  $\text{Deg}(\mathcal{E})$  (cf. [8], 3-). Let  $\mathcal{O}_Y(1)$  be a very ample line bundle on  $Y$ . Then the Hilbert polynomial of  $\mathcal{E}$  is

$$P_{\mathcal{E}}(m) = R(\mathcal{E}) \deg(\mathcal{O}_C(1))m + \text{Deg}(\mathcal{E}) + R(\mathcal{E})(1 - g)$$

(where  $g$  is the genus of  $C$ ).

We deduce from proposition 2.1.1:

**2.3.1. Proposition:** *Let  $Y$  be a multiple curve with support  $C$ . Then  $Y$  is a primitive multiple curve if and only if  $\mathcal{I}_C/\mathcal{I}_C^2$  is zero, or a line bundle on  $C$ .*

**2.3.2. Parametrization of double curves** - In the case of double curves, D. Bayer and D. Eisenbud have obtained in [2] the following classification: if  $Y$  is of multiplicity 2, we have an exact sequence of vector bundles on  $C$

$$0 \longrightarrow L \longrightarrow \Omega_{Y|C} \longrightarrow \omega_C \longrightarrow 0$$

which is split if and only if  $Y$  is the *trivial curve*, i.e. the second infinitesimal neighbourhood of  $C$ , embedded by the zero section in the dual bundle  $L^*$ , seen as a surface. If  $Y$  is not trivial, it is completely determined by the line of  $\text{Ext}_{\mathcal{O}_C}^1(\omega_C, L)$  induced by the preceding exact sequence. The non trivial primitive curves of multiplicity 2 and of associated line bundle  $L$  are therefore parametrized by the projective space  $\mathbb{P}(\text{Ext}_{\mathcal{O}_C}^1(\omega_C, L))$ .

#### 2.4. SIMPLE PRIMITIVE MULTIPLE CURVES

Let  $C$  be a smooth projective irreducible curve,  $n \geq 2$  an integer and  $C_n$  a primitive multiple curve of multiplicity  $n$  and associated reduced curve  $C$ . Then the ideal sheaf  $\mathcal{I}_C$  of  $C$  in  $C_n$  is a line bundle on  $C_{n-1}$ .

We say that  $C_n$  is *simple* si  $\mathcal{I}_C \simeq \mathcal{O}_{n-1}$ .

In this case the line bundle on  $C$  associated to  $C_n$  is  $\mathcal{O}_C$ . The following result is proved in [10] (théorème 1.2.1):

**2.4.1. Theorem:** *Suppose that  $C_n$  is simple. Then there exists a flat family of smooth projective curves  $\tau : \mathcal{C} \rightarrow \mathbb{C}$  such that  $\tau^{-1}(0) \simeq C$  and that  $C_n$  is isomorphic to the  $n$ -th infinitesimal neighbourhood of  $C$  in  $\mathcal{C}$ .*

### 3. REDUCIBLE REDUCED DEFORMATIONS OF PRIMITIVE MULTIPLES CURVES

#### 3.1. CONNECTED COMPONENTS

Let  $(S, P)$  be a germ of smooth curve and  $t \in \mathcal{O}_{S,P}$  a generator of the maximal ideal. Let  $n > 0$  be an integer and  $Y = C_n$  a projective primitive multiple curve of multiplicity  $n$ .

Let  $k > 0$  be an integer. Let  $\pi : \mathcal{C} \rightarrow S$  be a flat morphism, where  $\mathcal{C}$  is a reduced algebraic variety, such that

- For every closed point  $s \in S$  such that  $s \neq P$ , the fiber  $\mathcal{C}_s$  has  $k$  irreducible components, which are smooth and transverse, and any three of these components have no common point.
- The fiber  $\mathcal{C}_P$  is isomorphic to  $C_n$ .

It is easy to see that the irreducible components of  $\mathcal{C}$  are reduced surfaces.

Let  $Z$  be the open subset of  $\mathcal{C} \setminus \mathcal{C}_P$  of points  $z$  belonging to only one irreducible component of  $\mathcal{C}_{\pi(z)}$ . Then the restriction of  $\pi : Z \rightarrow S \setminus \{P\}$  is a smooth morphism. For every  $s \in S \setminus \{P\}$ , let  $\mathcal{C}'_s = \mathcal{C}_s \cap Z$ . It is the open subset of smooth points of  $\mathcal{C}_s$ .

Let  $z \in Z$  and  $s = \pi(z)$ . There exists a neighbourhood (for the usual topology)  $U$  of  $s$ , isomorphic to  $\mathbb{C}$ , and a neighbourhood  $V$  of  $z$  such that  $V \simeq \mathbb{C}^2$ ,  $\pi(V) = U$ , the restriction of



$\pi : V \rightarrow U$  being the projection  $\mathbb{C}^2 \rightarrow \mathbb{C}$  on the first factor. We deduce easily from that the following facts:

- let  $s \in S \setminus \{P\}$  and  $C_1$  an irreducible component of  $\mathcal{C}_s$ . Let  $z_1, z_2 \in C_1 \cap Z$ . Then there exists neighbourhoods (in  $Z$ , for the usual topology)  $U_1, U_2$  of  $z_1, z_2$  respectively, such that if  $y_1 \in U_1, y_2 \in U_2$  are such that  $\pi(y_1) = \pi(y_2)$ , then  $y_1$  and  $y_2$  belong to the same irreducible component of  $\mathcal{C}_{\pi(y_1)}$ .
- for every continuous map  $\sigma : [0, 1] \rightarrow S \setminus \{P\}$  and every  $z \in Z$  such that  $\sigma(0) = \pi(z)$  there exists a lifting of  $\sigma$ ,  $\sigma' : [0, 1] \rightarrow Z$  such that  $\sigma'(0) = z$ . Moreover, if  $\sigma'' : [0, 1] \rightarrow Z$  is another lifting of  $\sigma$  such that  $\sigma''(0) = z$ , then  $\sigma'(1)$  and  $\sigma''(1)$  are in the same irreducible component of  $\mathcal{C}_{\sigma(1)}$ . More generally, if we only impose that  $\sigma''(0)$  is in the same irreducible component of  $\mathcal{C}_{\sigma(0)}$  as  $z$ , then  $\sigma'(1)$  and  $\sigma''(1)$  are in the same irreducible component of  $\mathcal{C}_{\sigma(1)}$ .

**3.1.1. Lemma:** *Let  $\sigma_0, \sigma_1 : [0, 1] \rightarrow S \setminus \{P\}$  be two continuous maps such that  $\sigma_0(0) = \sigma_1(0)$ ,  $s = \sigma_0(1) = \sigma_1(1)$ . Suppose that they are homotopic. Let  $\sigma'_0, \sigma'_1$  be liftings  $[0, 1] \rightarrow Z$  of  $\sigma_0, \sigma_1$  respectively, such that  $\sigma'_0(0) = \sigma'_1(0)$ . Then  $\sigma'_0(1)$  and  $\sigma'_1(1)$  belong to the same irreducible component of  $\mathcal{C}'_s$ .*

*Proof.* Let

$$\Psi : [0, 1] \times [0, 1] \longrightarrow S \setminus \{P\}$$

be an homotopy:

$$\Psi(0, t) = \sigma_0(t), \quad \Psi(1, t) = \sigma_1(t), \quad \Psi(t, 0) = \sigma_0(0), \quad \Psi(t, 1) = \sigma_0(1)$$

for  $0 \leq t \leq 1$ . For every  $u \in [0, 1]$  and  $\epsilon > 0$  let  $I_{u, \epsilon} = [u - \epsilon, u + \epsilon] \cap [0, 1]$ . By using the local structure of  $\pi|_Z$  for the usual topology it is easy to see that for every  $u \in [0, 1]$ , there exists an  $\epsilon > 0$  such that the restriction of  $\Psi$

$$I_{u, \epsilon} \times [0, 1] \longrightarrow S \setminus \{P\}$$

can be lifted to a morphism

$$\Psi' : I_{u, \epsilon} \times [0, 1] \longrightarrow Z$$

such that  $\Psi'(t, 0) = \sigma'_0(0)$  for every  $t \in I_{u, \epsilon}$ . It follows that if  $I_{u, \epsilon} = [a_{u, \epsilon}, b_{u, \epsilon}]$ , then  $\Psi'(a_{u, \epsilon}, 1)$  and  $\Psi'(b_{u, \epsilon}, 1)$  are in the same irreducible component of  $\mathcal{C}'_{\sigma_0(1)}$ . Now we have just to cover  $[0, 1]$  with a finite number of intervals  $I_{u, \epsilon}$  to obtain the result.  $\square$

Let  $s \in S \setminus \{P\}$ ,  $D_1, \dots, D_k$  be the irreducible components of  $\mathcal{C}'_s$  and  $x_i \in D_i$  for  $1 \leq i \leq k$ . Let  $\sigma$  be a loop of  $S \setminus \{P\}$  with origin  $s$ , defining a generator of  $\pi_1(S \setminus \{P\})$ . Let  $i$  be an integer such that  $1 \leq i \leq k$ . The liftings  $\sigma' : [0, 1] \rightarrow Z$  of  $\sigma$  such that  $\sigma'(0) = x_i$  end up at a component  $D_j$  which does not depend on  $x_i$ . Hence we can write

$$j = \alpha_{\mathcal{C}}(i).$$

**3.1.2. Lemma:**  $\alpha_{\mathcal{C}}$  is a permutation of  $\{1, \dots, k\}$ .

*Proof.* Suppose that  $i \neq j$  and  $\alpha_{\mathcal{C}}(i) = \alpha_{\mathcal{C}}(j)$ . By inverting the paths we find liftings of paths from  $D_{\alpha_{\mathcal{C}}(i)}$  to  $D_i$  and  $D_j$ . This contradicts lemma 3.1.1.  $\square$

Let  $p > 0$  be an integer such that  $\alpha_{\mathcal{C}}^p = I_{\{1, \dots, k\}}$ . Let  $t$  be a generator of the maximal ideal of  $\mathcal{O}_{S,P}$ ,  $K$  the field of rational functions on  $S$  and  $K' = K(t^{1/p})$ . Let  $S'$  be the germ of curve corresponding to  $K'$ ,  $\theta : S' \rightarrow S$  canonical the morphism and  $P'$  the unique point of  $\theta^{-1}(P)$ . Let  $\mathcal{D} = \theta^*(\mathcal{C})$ . We have therefore a cartesian diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\rho} & S' \\ \downarrow \Theta & & \downarrow \theta \\ \mathcal{C} & \xrightarrow{\pi} & S \end{array}$$

where  $\rho$  is flat, and for every  $s' \in S'$ ,  $\Theta$  induces an isomorphism  $\mathcal{D}_{s'} \simeq \mathcal{C}_{\theta(s')}$ . We have

$$\alpha_{\mathcal{D}} = I_{\{1, \dots, k\}}.$$

Let  $Z' \subset \mathcal{D}$  be the complement of the union of  $\rho^{-1}(P')$  and of the singular points of the curves  $\mathcal{D}_{s'}$ ,  $s' \neq P'$  (hence  $Z' = \Theta^{-1}(Z)$ ).

**3.1.3. Proposition:** *The open subset  $Z'$  has exactly  $k$  irreducible components  $Z'_1, \dots, Z'_k$ . Let  $\overline{Z'_1}, \dots, \overline{Z'_k}$  be their closures in  $\mathcal{D}$ . Then for every  $s' \in S' \setminus \{P'\}$ , the  $Z'_i \cap \mathcal{D}_{s'}$ ,  $1 \leq i \leq k$ , are the irreducible components of  $\mathcal{D}_{s'}$  minus the intersection points with the other components, and the  $\overline{Z'_i} \cap \mathcal{D}_{s'}$  are the irreducible components of  $\mathcal{D}_{s'}$ .*

**3.1.4. Definition:** *Let  $k > 0$  be an integer. We call reducible deformation of length  $k$  of  $C_n$  a flat morphism  $\pi : \mathcal{C} \rightarrow S$ , where  $\mathcal{C}$  is a reduced algebraic variety, such that*

- *For every closed point  $s \in S$ ,  $s \neq P$ , the fiber  $\mathcal{C}_s$  has  $k$  irreducible components, which are smooth and transverse, and any three of these components have no common point.*
- *The fiber  $\mathcal{C}_P$  is isomorphic to  $C_n$ .*
- *We have  $\alpha_{\mathcal{C}} = I_{\{1, \dots, k\}}$ .*

### 3.2. MAXIMAL REDUCIBLE DEFORMATIONS

Let  $(S, P)$  be a germ of smooth curve and  $t \in \mathcal{O}_{S,P}$  a generator of the maximal ideal. Let  $n > 0$  be an integer and  $Y = C_n$  a projective primitive multiple curve of multiplicity  $n$ , with underlying smooth curve  $C$ . We note  $g$  the genus of  $C$  and  $L$  the line bundle on  $C$  associated to  $C_n$ .

Let  $\pi : \mathcal{C} \rightarrow S$  be a reducible deformation of length  $k$  of  $C_n$ . Let  $Z_1, \dots, Z_k$  be the closed subvarieties of  $\pi^{-1}(S \setminus \{P\})$  such that for every  $s \in S \setminus \{P\}$ ,  $Z_{1s}, \dots, Z_{ks}$  are the irreducible components of  $\mathcal{C}_s$  (cf. prop. 3.1.3).

For  $1 \leq i \leq k$ , we denote by  $\mathcal{J}_i$  the ideal sheaf of  $Z_1 \cup \dots \cup Z_i$  in  $\pi^{-1}(S \setminus \{P\})$ . This sheaf is flat on  $S \setminus \{P\}$ , and we have

$$0 = \mathcal{J}_k \subset \mathcal{J}_{k-1} \subset \dots \subset \mathcal{J}_1 \subset \mathcal{O}_{\pi^{-1}(S \setminus \{P\})}.$$

The quotients  $\mathcal{O}_{\pi^{-1}(S \setminus \{P\})} / \mathcal{J}_1$ ,  $\mathcal{J}_i / \mathcal{J}_{i+1}$ ,  $1 \leq i < k$ , are also flat on  $S \setminus \{P\}$ . We obtain the filtration of sheaves on  $\mathcal{C}$

$$0 = \overline{\mathcal{J}_k} \subset \overline{\mathcal{J}_{k-1}} \subset \dots \subset \overline{\mathcal{J}_1} \subset \mathcal{O}_{\mathcal{C}}.$$

(cf. 2.2). According to proposition 2.2.2 the quotients  $\mathcal{O}_C/\overline{\mathcal{J}}_1$  and  $\overline{\mathcal{J}}_i/\overline{\mathcal{J}}_{i+1}$ ,  $1 \leq i < n$ , are flat on  $S$ . We have  $\mathcal{O}_{\pi^{-1}(S \setminus \{P\})}/\overline{\mathcal{J}}_1 = \mathcal{O}_{Z_1}$ . We denote by  $\mathbf{X}_i$  the closed subvariety of  $\mathcal{C}$  corresponding to the ideal sheaf  $\overline{\mathcal{J}}_i$ .

Similarly we consider the ideal sheaf  $\mathcal{J}'_i$  of  $Z_{i+1} \cup \dots \cup Z_n$  on  $\pi^{-1}(S \setminus \{P\})$ , the associated ideal sheaf  $\overline{\mathcal{J}}'_i$  on  $\mathcal{C}$  and the corresponding subvariety  $\mathbf{X}'_i$ .

**3.2.1. Proposition:** *We have  $k \leq n$ .*

*Proof.* Let  $\mathcal{E}_0 = \mathcal{O}_C/\overline{\mathcal{J}}_1$  and  $\mathcal{E}_i = \overline{\mathcal{J}}_i/\overline{\mathcal{J}}_{i+1}$  for  $1 \leq i < n$ . The sheaves  $\mathcal{E}_{iP}$  are not concentrated on a finite number of points. To see this we use a very ample line bundle  $\mathcal{O}(1)$  on  $\mathcal{C}$ . The Hilbert polynomial of  $\mathcal{E}_{iP}$  is the same as that of  $\mathcal{E}_{is}$ ,  $s \neq P$ , hence it is not constant. So we have  $R(\mathcal{E}_i) \geq 1$  (cf. 2.3), and since

$$(1) \quad n = R(\mathcal{O}_{C_n}) = \sum_{i=0}^k R(\mathcal{E}_{iP}),$$

we have  $k \leq n$ . □

**3.2.2. Definition:** *We say that  $\pi$  (ou  $\mathcal{C}$ ) is a maximal reducible deformation of  $C_n$  if  $k = n$ .*

**3.2.3. Theorem:** *Suppose that  $\mathcal{C}$  is a maximal reducible deformation of  $C_n$ . Then we have, for  $1 \leq i < n$*

$$\overline{\mathcal{J}}_{i,P} = \mathcal{I}_{C_i, C_n}$$

*and  $\mathbf{X}_i$  is a maximal reducible deformation of  $C_i$ .*

*Proof.* Let  $\mathcal{O}_C(1)$  be a very ample line bundle on  $\mathcal{C}$ .

Let  $Q$  be a closed point of  $C$ . Let  $z \in \mathcal{O}_{n,Q}$  be an equation of  $C$  and  $x \in \mathcal{O}_{n,Q}$  over a generator of the maximal ideal of  $Q$  in  $\mathcal{O}_{C,Q}$ . Let  $\mathbf{z}, \mathbf{x} \in \mathcal{O}_{C,Q}$  be over  $z, x$  respectively. The maximal ideal of  $\mathcal{O}_{n,Q}$  is  $(x, z)$ . The maximal ideal of  $\mathcal{O}_{C,Q}$  is generated by  $\mathbf{z}, \mathbf{x}, t$ . It follows from proposition 2.1.1 that there exists a neighbourhood  $U$  of  $Q$  in  $\mathcal{C}$  and an embedding  $j : U \rightarrow \mathbb{P}_3$ . We can assume that the restriction of  $j$  to  $\overline{Z}_1 \cap U$  is induced by the morphism  $\phi : \mathbb{C}[X, Z, T] \rightarrow \mathcal{O}_{\overline{Z}_1, Q}$  of  $\mathbb{C}$ -algebras which associates  $x, z, t$  to  $X, Z, T$  respectively.

Since  $\mathcal{C}$  is reduced,  $U$  is an open subset of a reduced hypersurface of  $\mathbb{P}_3$  having  $n$  irreducible components, corresponding to  $\overline{Z}_1, \dots, \overline{Z}_n$ . It is then clear that  $\mathbf{X}_i$ , being the smallest subscheme of  $\mathcal{C}$  containing  $Z_1 \setminus C, \dots, Z_i \setminus C$ , is the union in  $U$  of the first  $i$  hypersurface components.

Since  $j(\overline{Z}_1)$  is an hypersurface, the kernel of  $\phi$  is a principal ideal generated by the equation  $F$  of the image of  $Z_1$ .

Recall that  $\mathcal{O}_n = \mathcal{O}_{C_n} = (\mathcal{O}_C)_P$ . We have  $R(\mathcal{O}_n/\overline{\mathcal{J}}_{1,P}) = 1$  according to (1). Hence there exists a nonempty open subset  $V$  of  $C_n$  such that  $(\mathcal{O}_n/\overline{\mathcal{J}}_{1,P})|_V$  is a line bundle on  $V \cap C$ . It follows that the projection  $\mathcal{O}_n \rightarrow \mathcal{O}_C$  vanishes on  $\overline{\mathcal{J}}_{1,P}|_V$ . Since  $\mathcal{O}_C$  is torsion free this projection vanishes everywhere on  $\overline{\mathcal{J}}_1$ , i.e.  $\overline{\mathcal{J}}_{1P} \subset \mathcal{I}_{C, C_n}$ , with equality on  $V$ .

The sheaf  $\mathcal{E}_0 = \mathcal{O}_C/\overline{\mathcal{J}}_1$  is the structural sheaf of  $\overline{Z}_1$ , and the projection  $\overline{Z}_1 \rightarrow S$  is a flat morphism. For every  $s \in S \setminus \{P\}$ ,  $(\overline{Z}_1)_s$  is a smooth curve. The fiber  $(\overline{Z}_1)_P$  consists of  $C$  and a finite number of embedded points. There exists flat families of curves whose general fiber is smooth

and the special fiber consists of an integral curve and some embedded points (cf. [15], III, Example 9.8.4). We will show that this cannot happen in our case, i.e. we have  $\overline{\mathcal{J}}_{1P} = \mathcal{I}_{C,C_n}$ .

Let  $\mathbf{m} = (X, Z, T) \subset \mathbb{C}[X, Z, T]$ , and  $\mathbf{m}_{Z_1}$  the maximal ideal of  $\mathcal{O}_{\overline{Z_1},Q}$ . The ideal of  $(\overline{Z_1})_P$  in  $\mathcal{O}_{n,Q}$  contains  $z^q$  and  $x^p z$  (for suitable minimal integers  $p \geq 0, q > 0$ ), with  $p > 0$  if and only if  $Q$  is an embedded point. Hence the ideal of  $\overline{Z_1}$  in  $\mathcal{O}_{C,Q}$  contains elements of type  $\mathbf{x}^p \mathbf{z} - t\alpha$ ,  $\mathbf{z}^q - t\beta$ , with  $\alpha, \beta \in \mathcal{O}_{C,Q}$ .

Let  $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$  be the completion of  $\mathcal{O}_{\overline{Z_1},Q}$  with respect to  $\mathbf{m}_{Z_1}$  and

$$\widehat{\phi} : \mathbb{C}((X, Z, T)) \longrightarrow \widehat{\mathcal{O}_{\overline{Z_1},Q}}$$

the morphism deduced from  $\phi$ . We can also see  $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$  as the completion with respect to  $(X, Z, T)$  of  $\mathcal{O}_{\overline{Z_1},Q}$  seen as a  $\mathbb{C}[X, Z, T]$ -module. It follows that  $\ker(\widehat{\phi}) = (F)$  (cf. [11], lemma 7.15). Note that  $\widehat{\phi}$  is surjective (this is why we use completions). Let  $\alpha, \beta \in \mathbb{C}((X, Y, Z))$  be such that  $\widehat{\phi}(\alpha) = \alpha$ ,  $\widehat{\phi}(\beta) = \beta$ . So we have

$$X^p Z - T\alpha, Z^q - T\beta \in \ker(\widehat{\phi}).$$

Hence there exists  $A, B \in \mathbb{C}((X, Z, T))$  such that  $X^p Z - T\alpha = AF$ ,  $Z^q - T\beta = BF$ . We can write in a unique way

$$A = A_0 + TA_1, B = B_0 + TB_1, F = F_0 + TF_1,$$

with  $A_0, B_0, F_0 \in \mathbb{C}((X, Z))$  and  $A_1, B_1, F_1 \in \mathbb{C}((X, Z, T))$ , and we have

$$A_0 F_0 = X^p Z, B_0 F_0 = Z^q.$$

Since  $F$  is not invertible, it follows that  $F_0$  is of the form  $F_0 = cZ$ , with  $c \in \mathbb{C}((X, Z, T))$  invertible. So we have  $F = cZ + TF_1$ . It follows that  $z \in (t)$  in  $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$ . This implies that this is also true in  $\mathcal{O}_{\overline{Z_1},Q}$  : in fact the assertion in  $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$  implies that

$$z \in \bigcap_{n \geq 0} ((t) + \mathbf{m}_{Z_1}^n)$$

in  $\mathcal{O}_{\overline{Z_1},Q}$ , and the latter is equal to  $(t)$  according to [18], vol. II, chap. VIII, theorem 9. Hence  $z \in (t)$  in  $\mathcal{O}_{\overline{Z_1},Q}$ , i.e.  $p = 0$  and  $Q$  is not an embedded point. So there are no embedded points. This implies that  $\overline{\mathcal{J}}_{1P} = \mathcal{I}_{C,C_n}$ . Similarly, if  $I_j$  denotes the ideal sheaf of  $\overline{Z_j}$  for  $1 \leq j \leq n$ , we have  $I_{j,P} = \mathcal{I}_{C,C_n}$ . Since the restriction of  $\pi : \overline{Z_j} \rightarrow S$  is flat, the curves  $\mathcal{E}_{j,s}$ ,  $s \neq P$ , have the same genus as  $C$ , and the same Hilbert polynomial with respect to  $\mathcal{O}_C(1)$ .

Now we show that  $\mathbf{X}'_1$  is a maximal reducible deformation of  $C_{n-1}$ . We need only to show that  $\mathbf{X}'_{1,P} = C_{n-1}$ . As we have seen, for  $2 \leq j \leq n$ , a local equation of  $\overline{Z_j}$  at any point  $Q \in C$  induces a generator  $u_j$  of  $\mathcal{I}_{C,C_n,Q}$ . Hence  $u = \prod_{2 \leq j \leq n} u_j$  is a generator of  $\mathcal{I}_{C_{n-1},C_n,Q}$ . But  $u = 0$  on  $\mathbf{X}'_1$ . It follows that  $\mathbf{X}'_{1,P} \subset C_{n-1}$ . But the Hilbert polynomial of  $\mathcal{O}_{C_{n-1}}$  is the same as that of the structural sheaves of the fibers of the flat morphism  $\mathbf{X}'_1 \rightarrow S$  over  $s \neq P$ , hence the same as  $\mathcal{O}_{\mathbf{X}'_{1,P}}$ . Hence  $\mathbf{X}'_{1,P} = C_{n-1}$ .

The theorem 3.2.3 is then easily proved by induction on  $n$ . □

**3.2.4. Corollary :** *Let  $s \in S \setminus \{P\}$  and  $D_1, D_2$  be two irreducible components of  $\mathcal{C}_s$ . Then  $D_1$  is of genus  $g$  and  $D_1 \cap D_2$  consists of  $-\deg(L)$  points.*

*Proof.* According to theorem 3.2.3, there exists a flat family of smooth curves  $\mathbf{C}$  parametrized by  $S$  such that  $\mathbf{C}_P = C$  and  $\mathbf{C}_s = D_1$ . So the genus of  $D_1$  is equal to that of  $C$ .

Let us prove the second assertion. Again according to theorem 3.2.3 we can suppose that  $n = 2$ . We have then  $\chi(\mathcal{C}_s) = \chi(C_2) = 2\chi(C) + \deg(L)$ . Let  $x_1, \dots, x_N$  be the intersection points of  $D_1$  and  $D_2$ . We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{D_2}(-x_1 - \dots - x_N) \longrightarrow \mathcal{O}_{\mathcal{C}_s} \longrightarrow \mathcal{O}_{D_1} \longrightarrow 0.$$

Whence  $\chi(\mathcal{O}_{\mathcal{C}_s}) = \chi(D_1) + \chi(D_2) - N = 2\chi(\mathcal{O}_C) - N$  (according to the first assertion). Whence  $N = -\deg(L)$ .  $\square$

**3.2.5.** It follows from the previous results that if  $\pi : \mathcal{C} \rightarrow S$  is a maximal reducible deformation of  $C_n$ , then we have

- (i)  $\deg(L) \leq 0$ .
- (ii)  $\mathcal{C}$  has exactly  $n$  irreducible components  $\mathcal{C}_1, \dots, \mathcal{C}_n$ .
- (iii) For  $1 \leq i \leq n$ , the restriction of  $\pi$ ,  $\pi_i : \mathcal{C}_i \rightarrow S$  is a flat morphism, and  $\pi_i^{-1}(P) = C$ .
- (iv) For every nonempty subset  $I \subset \{1, \dots, n\}$ , let  $\mathcal{C}_I$  be the union of the  $\mathcal{C}_i$  such that  $i \in I$ , and  $m$  the number of elements of  $I$ . Then the restriction of  $\pi$ ,  $\pi_I : \mathcal{C}_I \rightarrow S$  is a maximal reducible deformation of  $C_m$ .

The following is immediate, and shows that we need only to consider maximal reducible deformations parametrized by a neighbourhood of 0 in  $\mathbb{C}$ :

**3.2.6. Proposition:** *Let  $t \in \mathcal{O}_S(P)$  be a generator of the maximal ideal, and  $\pi : \mathcal{C} \rightarrow S$  a maximal reducible deformation of  $C_n$ . Let  $S' \subset S$  an open neighbourhood of  $P$  where  $t$  is defined and  $\mathcal{C}' = \pi^{-1}(U)$ ,  $V = t(U)$ . Then  $\pi' = t \circ \pi : \mathcal{C}' \rightarrow V$  is a maximal reducible deformation of  $C_n$ .*

## 4. FRAGMENTED DEFORMATIONS OF PRIMITIVE MULTIPLE CURVES

The fragmented deformations of primitive multiple curves are particular cases of reducible deformations.

In this chapter  $(S, P)$  denotes a germ of smooth curve. Let  $t \in \mathcal{O}_{S, P}$  be a generator of the maximal ideal of  $P$ . We can suppose that  $t$  is defined on the whole of  $S$ , and that the ideal sheaf of  $P$  in  $S$  is generated by  $t$ .

### 4.1. FRAGMENTED DEFORMATIONS AND GLUEING

Let  $n > 0$  be an integer and  $Y = C_n$  a projective primitive multiple curve of multiplicity  $n$ .

**4.1.1. Definition:** *Let  $k > 0$  be an integer. A general fragmented deformation of length  $k$  of  $C_n$  is a flat morphism  $\pi : \mathcal{C} \rightarrow S$  such that for every point  $s \neq P$  of  $S$ , the fiber  $\mathcal{C}_s$  is a disjoint union of  $k$  projective smooth irreducible curves, and such that  $\mathcal{C}_P$  is isomorphic to  $C_n$ .*

We have then  $k \leq n$ . If  $k = n$  we say that  $\pi$  (or  $\mathcal{C}$ ) is a *general maximal fragmented deformation* of  $C_n$ . We suppose in the sequel that it is the case.

The line bundle on  $C$  associated to  $C_n$  is  $\mathcal{O}_C$  (by proposition 3.2.4).

Let  $p > 0$  be an integer. Let  $K$  be the field of rational functions on  $S$  and  $K' = K(t^{1/p})$ . Let  $S'$  be the germ of curve corresponding to  $K'$ ,  $\theta : S' \rightarrow S$  the canonical morphism and  $P'$  the unique point of  $\theta^{-1}(P)$ . Let  $\mathcal{D} = \theta^*(\mathcal{C})$ . So we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\rho} & S' \\ \downarrow \Theta & & \downarrow \theta \\ \mathcal{C} & \xrightarrow{\pi} & S \end{array}$$

where  $\rho$  is flat, and for every  $s' \in S'$ ,  $\Theta$  induces an isomorphism  $\mathcal{D}_{s'} \simeq \mathcal{C}_{\theta(s')}$ .

**4.1.2. Proposition:** *For a suitable choice of  $p$ ,  $\mathcal{D}$  has exactly  $n$  irreducible components  $\mathcal{D}_1, \dots, \mathcal{D}_n$ , and for every point  $s \neq P'$  of  $S'$ ,  $\mathcal{D}_{1s}, \dots, \mathcal{D}_{ns}$  are the irreducible components of  $\mathcal{D}_s$ , for  $1 \leq i \leq n$  the restriction of  $\rho : \mathcal{D}_{is} \rightarrow S'$  is flat, and  $\mathcal{D}_{P'} = C_n$ .*

(See proposition 3.1.3)

**4.1.3. Definition:** *A fragmented deformation of  $C_n$  is a general maximal fragmented deformation of length  $n$  of  $C_n$  having  $n$  irreducible components.*

We suppose in the sequel that  $\mathcal{C}$  is a fragmented deformation of  $C_n$ , union of  $n$  irreducible components  $\mathcal{C}_1, \dots, \mathcal{C}_n$ .

**4.1.4. Proposition:** *Let  $I \subset \{1, \dots, n\}$  a nonempty subset having  $m$  elements. Let  $\mathcal{C}_I = \cup_{i \in I} \mathcal{C}_i$ . Then the restriction of  $\pi, \mathcal{C}_I \rightarrow S$ , is flat, and the fiber  $\mathcal{C}_{IP}$  is canonically isomorphic to  $C_m$ .*

(See 3.2.5)

In particular there exists a filtration of ideal sheaves

$$0 \subset \mathcal{I}_1 \subset \dots \subset \mathcal{I}_{n-1} \subset \mathcal{O}_{\mathcal{C}}$$

such that for  $1 \leq i < n$  and  $s \in S \setminus \{P\}$ ,  $\mathcal{I}_{is}$  is the ideal sheaf of  $\cup_{j=i}^n \mathcal{C}_{js}$ , and that  $\mathcal{I}_{iP}$  is that of  $C_{n-i}$ .

**4.1.5. Definition:** *For  $1 \leq i \leq n$ , let  $\pi : \mathcal{C}_i \rightarrow S$  be a flat family of smooth projective irreducible curves, with a fixed isomorphism  $\pi_i^{-1}(P) \simeq C$ . A glueing of  $\mathcal{C}_1, \dots, \mathcal{C}_n$  along  $C$  is an algebraic variety  $\mathcal{D}$  such that*

- for  $1 \leq i \leq n$ ,  $\mathcal{C}_i$  is isomorphic to a closed subvariety of  $\mathcal{D}$ , also denoted by  $\mathcal{C}_i$ , and  $\mathcal{D}$  is the union of these subvarieties.
- $\coprod_{1 \leq i \leq n} (\mathcal{C}_i \setminus C)$  is an open subset of  $\mathcal{D}$ .
- There exists a morphism  $\pi : \mathcal{D} \rightarrow S$  inducing  $\pi_i$  on  $\mathcal{C}_i$ , for  $1 \leq i \leq n$ .
- The subvarieties  $C = \pi_i^{-1}(P)$  of  $\mathcal{C}_i$  coincide in  $\mathcal{D}$ .

For example the previous fragmented deformation  $\mathcal{C}$  of  $C_n$  is a glueing of  $\mathcal{C}_1, \dots, \mathcal{C}_n$  along  $C$ . All the glueings of  $\mathcal{C}_1, \dots, \mathcal{C}_n$  along  $C$  have the same underlying Zariski topological space.

Let  $\mathcal{A}$  the *initial glueing* of the  $\mathcal{C}_i$  along  $C$ . It is an algebraic variety whose points are the same as those of  $\mathcal{C}$ , i.e.

$$(\prod_{i=1}^n \mathcal{C}_i) / \sim ,$$

where  $\sim$  is the equivalence relation: if  $x \in \mathcal{C}_i$  and  $y \in \mathcal{C}_j$ ,  $x \sim y$  if and only if  $x = y$ , or if  $x \in \mathcal{C}_{iP} \simeq C$ ,  $y \in \mathcal{C}_{jP} \simeq C$  and  $x = y$  in  $C$ . The structural sheaf is defined by : for every open subset  $U$  of  $\mathcal{A}$

$$\mathcal{O}_{\mathcal{A}}(U) = \{(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C}_1}(U \cap \mathcal{C}_1) \times \dots \times \mathcal{O}_{\mathcal{C}_n}(U \cap \mathcal{C}_n); \alpha_{1|C} = \dots = \alpha_{n|C}\}.$$

For every glueing  $\mathcal{D}$  of  $\mathcal{C}_1, \dots, \mathcal{C}_n$ , we have an obvious dominant morphism  $\mathcal{A} \rightarrow \mathcal{D}$ . It follows that the sheaf of rings  $\mathcal{O}_{\mathcal{D}}$  can be seen as a subsheaf of  $\mathcal{O}_{\mathcal{A}}$ .

The fiber  $D = \mathcal{A}_0$  is not a primitive multiple curve (if  $n > 2$ ): if  $\mathcal{I}_{C,D}$  denotes the ideal sheaf of  $C$  in  $D$  we have  $\mathcal{I}_{C,D}^2 = 0$ , and  $\mathcal{I}_{C,D} \simeq \mathcal{O}_C \otimes \mathbb{C}^{n-1}$ .

**4.1.6. Proposition:** *Let  $\mathcal{D}$  be a glueing of  $\mathcal{C}_1, \dots, \mathcal{C}_n$ . Then  $\pi^{-1}(P)$  is a primitive multiple curve if and only if for every closed point  $x$  of  $C$ , there exists a neighbourhood of  $x$  in  $\mathcal{D}$  that can be embedded in a smooth variety of dimension 3.*

*Proof.* Suppose that  $\pi^{-1}(P)$  is a primitive multiple curve. Then  $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$  is a principal module at  $x$ : suppose that the image of  $u \in m_{\mathcal{D},x}$  is a generator. The module  $m_{\mathcal{D},x}/\mathcal{I}_C$  is also principal (since it is the maximal ideal of  $x$  in  $C$ ): suppose that the image of  $v \in m_{\mathcal{D},x}$  is a generator. Then the images of  $u, v, \pi$  generate  $m_{\mathcal{D},x}/m_{\mathcal{D},x}^2$ , so according to proposition 2.1.1, we can locally embed  $\mathcal{D}$  in a smooth variety of dimension 3.

Conversely, suppose that a neighbourhood of  $x \in C$  in  $\mathcal{D}$  is embedded in a smooth variety  $Z$  of dimension 3. The proof of the fact that  $\pi^{-1}(P)$  is Cohen-Macaulay is similar to that of theorem 3.2.3. We can suppose that  $\pi$  is defined on  $Z$ . We have  $\pi|_{\mathcal{C}_1} = \pi_1 \notin m_{\mathcal{C}_1,x}^2$ , so  $\pi \notin m_{Z,x}^2$ . It follows that the surface of  $Z$  defined by  $\pi$  is smooth at  $x$ , and that we can locally embed  $\pi^{-1}(P)$  in a smooth surface. Hence  $\pi^{-1}(P)$  is a primitive multiple curve.  $\square$

## 4.2. FRAGMENTED DEFORMATIONS OF LENGTH 2

Let  $\pi : \mathcal{C} \rightarrow S$  be a fragmented deformation of  $C_2$ . So  $\mathcal{C}$  has two irreducible components  $\mathcal{C}_1, \mathcal{C}_2$ . Let  $\mathcal{A}$  be the glueing of  $\mathcal{C}_1$  and  $\mathcal{C}_2$  along  $C$ . For every open subset  $U$  of  $\mathcal{C}$ ,  $U$  is also an open subset of  $\mathcal{A}$  and  $\mathcal{O}_{\mathcal{C}}(U)$  is a sub-algebra of  $\mathcal{O}_{\mathcal{A}}(U)$ . For  $i = 1, 2$ , let  $\pi_i : \mathcal{C}_i \rightarrow S$  be the restriction of  $\pi$ . We will also denote  $t \circ \pi$  by  $\pi$ , and  $t \circ \pi_i$  by  $\pi_i$ . So we have  $\pi = (\pi_1, \pi_2) \in \mathcal{O}_{\mathcal{C}}(\mathcal{C})$ .

Let  $\mathcal{I}_C$  be the ideal sheaf of  $C$  in  $\mathcal{C}$ . Since  $C_2 = \pi^{-1}(P)$  we have  $\mathcal{I}_C^2 \subset \langle (\pi_1, \pi_2) \rangle$ .

Let  $m > 0$  be an integer,  $x \in C$ ,  $\alpha_1 \in \mathcal{O}_{\mathcal{C}_1,x}$ ,  $\alpha_2 \in \mathcal{O}_{\mathcal{C}_2,x}$ . We denote by  $[\alpha_1]_m$  (resp.  $[\alpha_2]_m$ ) the image of  $\alpha_1$  (resp.  $\alpha_2$ ) in  $\mathcal{O}_{\mathcal{C}_1,x}/(\pi_1^m)$  (resp.  $\mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^m)$ ).

**4.2.1. Proposition: 1** – *There exists an unique integer  $p > 0$  such that  $\mathcal{I}_C/\langle (\pi_1, \pi_2) \rangle$  is generated by the image of  $(\pi_1^p, 0)$ .*

**2** – The image of  $(0, \pi_2^p)$  generates  $\mathcal{I}_C / \langle (\pi_1, \pi_2) \rangle$ .

**3** – For every  $x \in C$ ,  $\alpha \in \mathcal{O}_{\mathcal{C}_1, x}$  and  $\beta \in \mathcal{O}_{\mathcal{C}_2, x}$ , we have  $(\pi_1^p \alpha, 0) \in \mathcal{O}_{\mathcal{C}, x}$  and  $(0, \pi_2^p \beta) \in \mathcal{O}_{\mathcal{C}, x}$ .

*Proof.* Let  $x \in C$  and  $u = (\pi_1 \alpha, \pi_2 \beta)$  whose image is a generator of  $\mathcal{I}_C / \langle (\pi_1, \pi_2) \rangle$  at  $x$  ( $\mathcal{I}_C / \langle (\pi_1, \pi_2) \rangle$  is a locally free sheaf of rank 1 of  $\mathcal{O}_C$ -modules). Let  $\beta_0 \in \mathcal{O}_{\mathcal{C}_1, x}$  be such that  $(\beta_0, \beta) \in \mathcal{O}_{\mathcal{C}, x}$ . Then the image of

$$u - (\pi_1, \pi_2)(\beta_0, \beta) = (\pi_1(\alpha - \beta_0), 0)$$

is also a generator of  $\mathcal{I}_C / \langle (\pi_1, \pi_2) \rangle$  at  $x$ . We can write it  $(\pi_1^p \lambda, 0)$ , where  $\lambda$  is not a multiple of  $\pi_1$ .

Now we show that  $p$  is the smallest integer  $q$  such that  $(\mathcal{I}_C / \langle (\pi_1, \pi_2) \rangle)_x$  contains the image of an element of the form  $(\pi_1^q \mu, 0)$ , with  $\mu$  not divisible by  $\pi_1$ . We can write

$$(\pi_1^q \mu, 0) = (u_1, u_2)(\pi_1^p \lambda, 0) + (v_1, v_2)(\pi_1, \pi_2)$$

with  $(u_1, u_2), (v_1, v_2) \in \mathcal{O}_{\mathcal{C}, x}$ . So we have  $v_2 = 0$ , hence  $(v_1, v_2) \in \mathcal{I}_{\mathcal{C}, x}$ . So we can write  $(v_1, v_2)$  as the sum of a multiple of  $(\pi_1^p \lambda, 0)$  and a multiple of  $(\pi_1, \pi_2)$ . Finally we obtain  $(\pi_1^q \mu, 0)$  as

$$(\pi_1^q \mu, 0) = (u_{12}, u_{22})(\pi_1^p \lambda, 0) + (v_{11}, 0)(\pi_1, \pi_2)^2.$$

In the same way we see that  $(\pi_1^q \mu, 0)$  can be written as

$$(\pi_1^q \mu, 0) = (u_{1p}, u_{2p})(\pi_1^p \lambda, 0) + (v_{1p}, 0)(\pi_1, \pi_2)^p,$$

which implies immediately that  $q \geq p$ .

It follows that  $p$  does not depend on  $x$  and that  $\mathcal{I}_C / \langle (\pi_1, \pi_2) \rangle$  is a subsheaf of  $\langle (\pi_1^p, 0) \rangle / \langle (\pi_1^{p+1}, 0) \rangle \simeq \mathcal{O}_C$ . Since  $\mathcal{I}_C / \langle (\pi_1, \pi_2) \rangle$  is of degree 0 (by corollary 3.2.4) it follows that  $\mathcal{I}_C / \langle (\pi_1, \pi_2) \rangle \simeq \langle (\pi_1^p, 0) \rangle / \langle (\pi_1^{p+1}, 0) \rangle$ , from which we deduce assertion 1- of proposition 4.2.1. The second assertion comes from the fact that  $(0, \pi_2^p) = \pi^p - (\pi_1^p, 0)$ .

To prove the third, we use the fact that there exists  $\alpha' \in \mathcal{O}_{\mathcal{C}_2, x}$  such that  $(\alpha, \alpha') \in \mathcal{O}_{\mathcal{C}, x}$  (because  $\mathcal{C}_1 \subset \mathcal{C}$ ). Hence  $(\pi_1^p, 0)(\alpha, \alpha') = (\pi_1^p \alpha, 0) \in \mathcal{O}_{\mathcal{C}, x}$ . Similarly, we obtain that  $(0, \pi_2^p \beta) \in \mathcal{O}_{\mathcal{C}, x}$ .  $\square$

According to the proof the proposition 4.2.1, for every  $x \in C$ ,  $p$  is the smallest integer  $q$  such that there exists an element of  $\mathcal{O}_{\mathcal{C}, x}$  of the form  $(\pi_1^q \alpha, 0)$  (resp.  $(0, \pi_2^q \alpha)$ ), with  $\alpha \in \mathcal{O}_{\mathcal{C}_1, x}$  (resp.  $\alpha \in \mathcal{O}_{\mathcal{C}_2, x}$ ) not vanishing on  $C$ .

Let  $x \in C$  and  $\alpha_1 \in \mathcal{O}_{\mathcal{C}_1, x}$ . Since  $\mathcal{C}_1 \subset \mathcal{C}$  there exists  $\alpha_2 \in \mathcal{O}_{\mathcal{C}_2, x}$  such that  $(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}, x}$ . Let  $\alpha'_2 \in \mathcal{O}_{\mathcal{C}_2, x}$  such that  $(\alpha_1, \alpha'_2) \in \mathcal{O}_{\mathcal{C}, x}$ . We have then  $(0, \alpha_2 - \alpha'_2) \in \mathcal{O}_{\mathcal{C}, x}$ . So there exists  $\alpha \in \mathcal{O}_{\mathcal{C}_2, x}$  such that  $\alpha_2 - \alpha'_2 = \pi_2^p \alpha$ . It follows that the image of  $\alpha_2$  in  $\mathcal{O}_{\mathcal{C}_2, x} / (\pi_2^p)$  is uniquely determined. Hence we have:

**4.2.2. Proposition:** *There exists a canonical isomorphism*

$$\Phi : \mathcal{C}_1^{(p)} \longrightarrow \mathcal{C}_2^{(p)}$$

*between the infinitesimal neighbourhoods of order  $p$  of  $\mathcal{C}_1$  and  $\mathcal{C}_2$  (i.e.  $\mathcal{O}_{\mathcal{C}_i}^{(p)} = \mathcal{O}_{\mathcal{C}_i} / (\pi_i^p)$ ), such that for every  $x \in C$ ,  $\alpha_1 \in \mathcal{O}_{\mathcal{C}_1, x}$  and  $\alpha_2 \in \mathcal{O}_{\mathcal{C}_2, x}$ , we have  $(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}, x}$  if and only if  $\Phi_x([\alpha_1]_p) = [\alpha_2]_p$ . For every  $\alpha \in \mathcal{O}_{\mathcal{C}_1, x}$  we have  $\Phi_x(\alpha)|_C = \alpha|_C$ , and  $\Phi_x(\pi_1) = \pi_2$ .*



The simplest case is  $p = 1$ . In this case  $\Phi : C \rightarrow C$  is the identity and  $\mathcal{C} = \mathcal{A}$  (the *initial* glueing).

**4.2.3. Converse** - Recall that  $\mathcal{A}$  denotes the initial glueing of  $\mathcal{C}_1, \mathcal{C}_2$  (cf. 4.1.5). Let  $\Phi : \mathcal{C}_1^{(p-1)} \rightarrow \mathcal{C}_2^{(p-1)}$  be an isomorphism inducing the identity on  $C$  and such that  $\Phi(\pi_1) = \pi_2$ . We define a subsheaf of algebras  $\mathcal{U}_\Phi$  of  $\mathcal{O}_\mathcal{A}$ :  $\mathcal{U}_\Phi = \mathcal{O}_\mathcal{A}$  on  $\mathcal{A} \setminus C$ , and for every point  $x$  of  $C$

$$\mathcal{U}_{\Phi,x} = \{(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} ; \Phi_x([\alpha_1]_p) = [\alpha_2]_p\}.$$

It is easy to see that  $\mathcal{U}_\Phi$  is the structural sheaf of an algebraic variety  $\mathcal{A}_\Phi$ , that the inclusion  $\mathcal{U}_\Phi \subset \mathcal{O}_\mathcal{A}$  defines a dominant morphism  $\mathcal{A} \rightarrow \mathcal{A}_\Phi$  inducing an isomorphism between the underlying topological spaces (for the Zariski topology), and that the composed morphisms  $\mathcal{C}_i \subset \mathcal{A} \rightarrow \mathcal{A}_\Phi$ ,  $i = 1, 2$ , are immersions. Moreover, the morphism  $\pi : \mathcal{A} \rightarrow S$  factorizes through  $\mathcal{A}_\Phi$  :

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A}_\Phi \xrightarrow{\pi_\Phi} S \\ & \searrow \pi & \nearrow \end{array}$$

and  $\pi_\Phi : \mathcal{A}_\Phi \rightarrow S$  is flat.

For  $2 \leq i \leq p$ , let  $\Phi^{(i)} : \mathcal{C}_1^{(i)} \rightarrow \mathcal{C}_2^{(i)}$  be the isomorphism induced by  $\Phi$ .

**4.2.4. Proposition:**  $\pi_\Phi^{-1}(P)$  is a primitive double curve.

*Proof.* Let  $x$  be a closed point of  $C$ . We first show that  $\mathcal{I}_{\mathcal{C},x}^2 \subset (\pi)$ . Let  $u = (\pi_1\alpha, \pi_2\beta) \in \mathcal{I}_{\mathcal{C},x}$ . Let  $\beta' \in \mathcal{O}_{\mathcal{C}_2,x}$  be such that  $\Phi_x([\alpha]_p) = [\beta']_p$ . We have then  $v = (\alpha, \beta') \in \mathcal{O}_{\mathcal{C},x}$ . We have  $u - \pi v = (0, \pi_2(\beta - \beta')) \in \mathcal{O}_{\mathcal{C},x}$ . Therefore  $[\pi_2(\beta - \beta')]_p = \Phi_x(0) = 0$ . Hence  $\pi_2(\beta - \beta') \in (\pi_2^p)$ . We can then write

$$u = \pi v + (0, \pi_2^p \gamma).$$

Let  $u' \in \mathcal{I}_{\mathcal{C},x}$ , that can be written as  $u' = \pi v' + (0, \pi_2^p \gamma')$ . We have then

$$uu' = \pi(\pi v v' + (0, \pi_2 \gamma')v + (0, \pi_2 \gamma)v') + (0, \pi_2^{2p-1} \gamma \gamma') \in (\pi).$$

It remains to show that  $\mathcal{I}_{\mathcal{C},x}/(\pi) \simeq \mathcal{O}_{\mathcal{C},x}$ . We have

$$\begin{aligned} \mathcal{I}_{\mathcal{C},x} &= \{(\pi_1\alpha, \pi_2\beta) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} ; \Phi_x([\pi_1\alpha]_p) = [\pi_2\beta]_p\} \\ &= \{(\pi_1\alpha, \pi_2\beta) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} ; \Phi_x^{(p-1)}([\alpha]_{p-1}) = [\beta]_{p-1}\}, \\ (\pi)_x &= \{(\pi_1\alpha, \pi_2\beta) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} ; \Phi_x([\alpha]_p) = [\beta]_p\}. \end{aligned}$$

So if  $(\pi_1\alpha, \pi_2\beta) \in \mathcal{I}_{\mathcal{C},x}$ , we have  $w = \Phi_x([\alpha]_p) - [\beta]_p \in (\pi_2^{p-1})_x / (\pi_2^p)_x \simeq \mathcal{O}_{\mathcal{C},x}$ . Hence we have a morphism of  $\mathcal{O}_{\mathcal{C},x}$ -modules

$$\begin{aligned} \lambda : \mathcal{I}_{\mathcal{C},x} &\longrightarrow \mathcal{O}_{\mathcal{C},x} \\ (\pi_1\alpha, \pi_2\beta) &\longmapsto w \end{aligned}$$

whose kernel is  $(\pi)_x$ . We have now only to show that  $\lambda$  is surjective, which follows from the fact that  $\lambda(\pi_1^p, 0) = 1$ .  $\square$

### 4.3. SPECTRUM OF A FRAGMENTED DEFORMATION AND IDEALS OF SUB-DEFORMATIONS

Let  $\pi : \mathcal{C} \rightarrow S$  be a fragmented deformation of  $C_n, \mathcal{C}_1, \dots, \mathcal{C}_n$  the irreducible components of  $\mathcal{C}$ . For  $1 \leq i \leq n$ , let  $\pi_i = \pi|_{\mathcal{C}_i}$ . As in 4.2, we denote also  $t \circ \pi_i$  by  $\pi_i$ . Let  $I = \{i, j\}$  be a subset of  $\{1, \dots, n\}$ , with  $i \neq j$ . Then  $\pi : \mathcal{C}_I \rightarrow S$  is a fragmented deformation of  $C_2$ . According to 4.2 there exists a unique integer  $p > 0$  such that  $\mathcal{I}_{\mathcal{C}, \mathcal{C}_I}/(\pi)$  is generated by the image of  $(\pi_i^p, 0)$  (and also by the image of  $(0, \pi_j^p)$ ). Recall that  $p$  is the smallest integer  $q$  such that  $\mathcal{I}_{\mathcal{C}, \mathcal{C}_I}$  contains a non zero element of the form  $(\pi_i^q \lambda, 0)$  (or  $(0, \pi_j^q \mu)$ ), with  $\lambda|_C \neq 0$  (resp.  $\mu|_C \neq 0$ ). Let

$$p_{ij} = p_{ji} = p,$$

and  $p_{ii} = 0$  for  $1 \leq i \leq n$ . The symmetric matrix  $(p_{ij})_{1 \leq i, j \leq n}$  is called the *spectrum* of  $\mathcal{C}$ .

**4.3.1. Generators of  $(\mathcal{I}_C^p + (\pi))/(\mathcal{I}_C^{p+1} + (\pi))$**  - Let  $i, j \in \{1, \dots, n\}$  be such that  $i \neq j$ . Let  $x \in C$ . Since  $\mathcal{C}_{\{i, j\}} \subset \mathcal{C}$  there exists an element  $\mathbf{u}_{ij} = (u_m)_{1 \leq m \leq n}$  of  $\mathcal{O}_{\mathcal{C}, x}$  such that  $u_i = 0$  and  $u_j = \pi_j^{p_{ij}}$ . According to proposition 4.1.4, the image of  $\mathbf{u}_{ij}$  generates  $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$  at  $x$ .

According to proposition 4.2.1 and the fact that the image of  $\mathbf{u}_{ij}$  generates  $\mathcal{I}_{\mathcal{C}, \mathcal{C}_{ij}, x}/(\mathcal{I}_{\mathcal{C}, \mathcal{C}_{ij}, x}^2 + (\pi))$ , for every integer  $m$  such that  $m \neq i, j$  and that  $1 \leq m \leq n$ ,  $u_m$  is of the form  $u_m = \alpha_{ij}^{(m)} \pi_m^{p_{im}}$ , with  $\alpha_{ij}^{(m)} \in \mathcal{O}_{\mathcal{C}_m, x}$  invertible. Let  $\alpha_{ij}^{(i)} = 0$  and  $\alpha_{ij}^{(j)} = 1$ .

**4.3.2. Proposition:**  $1 - \alpha_{ij|C}^{(m)}$  is a non zero constant, uniquely determined and independent of  $x$ .

**2 -** Let  $\mathbf{a}_{ij}^{(m)} = \alpha_{ij|C}^{(m)} \in \mathbb{C}$ . Then we have, for all integers  $i, j, k, m, q$  such that  $1 \leq i, j, k, m, q \leq n$ ,  $i \neq j$ ,  $i \neq k$

$$\mathbf{a}_{ik}^{(m)} \mathbf{a}_{ij}^{(q)} = \mathbf{a}_{ik}^{(q)} \mathbf{a}_{ij}^{(m)}.$$

In particular we have  $\mathbf{a}_{ij}^{(m)} = \mathbf{a}_{ik}^{(m)} \mathbf{a}_{ij}^{(k)}$  and  $\mathbf{a}_{ij}^{(m)} \mathbf{a}_{im}^{(j)} = 1$ .

*Proof.* Let  $\mathbf{u}'_{ij}$  having the same properties as  $\mathbf{u}_{ij}$ . Then  $\mathbf{v} = \mathbf{u}'_{ij} - \mathbf{u}_{ij} \in \mathcal{I}_{\mathcal{C}, x}^2 + (\pi)$ . So the image of  $\mathbf{v}$  in  $\mathcal{O}_{\mathcal{C}_{im}, x}$  belongs to  $\mathcal{I}_{\mathcal{C}, \mathcal{C}_{im}, x}^2 + (\pi)$ . It follows that the  $m$ -th component of  $\mathbf{v}$  is a multiple of  $\pi_m^{p_{im}+1}$ . Hence  $\alpha_{ij|C}^{(m)}$  is uniquely determined. It follows that when  $x$  varies the  $\alpha_{ij|C}^{(m)}$  can be glued together and define a global section of  $\mathcal{O}_C$ , which must be a constant. This proves 1-.

Now we prove 2-. There exists  $u \in \mathcal{O}_{\mathcal{C}, x}$  such that the  $k$ -th component of  $u$  is  $\alpha_{ij}^{(k)}$ , and  $u$  is invertible. Then the image of  $(v_m) = \frac{\mathbf{u}_{ij}}{u}$  generates  $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$ , and  $v_k = 1$ . Hence according to 1-, we have  $v_m|_C = \mathbf{a}_{ik}^{(m)}$ , i.e.

$$\frac{\mathbf{a}_{ij}^{(m)}}{\mathbf{a}_{ij}^{(k)}} = \mathbf{a}_{ik}^{(m)}.$$

We have the same equality with  $m$  instead of  $q$ , whence 2- is easily deduced.  $\square$

Let  $p$  an integer such that  $1 \leq p < n$ , and  $(i_1, j_1), \dots, (i_p, j_p)$   $p$  pairs of distinct integers of  $\{1, \dots, n\}$ . Then the image of  $\prod_{m=1}^p \mathbf{u}_{i_m j_m}$  is a generator of  $(\mathcal{I}_C^p + (\pi))/(\mathcal{I}_C^{p+1} + (\pi))$ .

Let  $I \subset \{1, \dots, n\}$  be a nonempty subset, distinct from  $\{1, \dots, n\}$ . Let  $i \in \{1, \dots, n\} \setminus I$ . Let

$$\mathbf{u}_{I,i} = \prod_{j \in I} \mathbf{u}_{ji}.$$

Recall that  $\mathcal{C}_I = \cup_{j \in I} \mathcal{C}_j \subset \mathcal{C}$ .

**4.3.3. Proposition:** *The ideal sheaf of  $\mathcal{C}_I$  is generated by  $\mathbf{u}_{I,i}$  at  $x$ .*

*Proof.* According to proposition 4.1.6 there exists an embedding of a neighbourhood of  $x$  in a smooth variety of dimension 3. In this variety each  $\mathcal{C}_i$  is a smooth surface defined by a single equation. The ideal of the union of the  $\mathcal{C}_i$ ,  $i \in I$  is the product of these equations.  $\square$

**4.3.4. Proposition:** *Let  $i, j, k$  be distinct integers such that  $1 \leq i, j, k \leq n$ . Then if  $p_{ij} < p_{jk}$ , we have  $p_{ik} = p_{ij}$ .*

*Proof.* We can come down to the case  $n = 3$  by considering  $\mathcal{C}_{\{i,j,k\}}$ . We can suppose that  $p_{23} \leq p_{12} \leq p_{13}$ , and we must show that  $p_{23} = p_{12}$ . We have

$$\mathbf{u}_{21} = (\pi_1^{p_{12}}, 0, \alpha_{21}^{(3)} \pi_3^{p_{23}}), \quad \mathbf{u}_{31} = (\pi_1^{p_{13}}, \alpha_{31}^{(2)} \pi_2^{p_{23}}, 0).$$

So

$$\mathbf{u}_{31} - \pi^{p_{13}-p_{12}} \mathbf{u}_{21} = (0, \alpha_{31}^{(2)} \pi_2^{p_{23}}, -\alpha_{21}^{(3)} \pi_3^{p_{23}+p_{13}-p_{12}}) \in \mathcal{O}_{\mathcal{C},x}.$$

Taking the image of this element in  $\mathcal{O}_{\mathcal{C}_{12},x}$ , we see that  $p_{23} \geq p_{12}$ , hence  $p_{23} = p_{12}$ .  $\square$

**4.3.5. Proposition: 1** – *Let  $i, j$  be distinct integers such that  $1 \leq i, j \leq n$ . Then we have  $\mathcal{I}_{\mathcal{C},x} = (\mathbf{u}_{ij}) + (\pi)$ .*

**2** – *Let  $v = (v_m)_{1 \leq m \leq n} \in \mathcal{I}_{\mathcal{C},x}$  such that  $v_i$  is a multiple of  $\pi_i^p$ , with  $p > 0$ . Then we have  $v \in (\mathbf{u}_{ij}) + (\pi^p)$ .*

*Proof.* Let  $N = 1 + \max_{1 \leq k \leq n} (q_i)$ . For every integer  $j$  such that  $1 \leq j \leq n$  we have  $(0, \dots, 0, \pi_j^{q_j}, 0, \dots, 0) \in \mathcal{O}_{\mathcal{C}}(\mathcal{C})$ . Hence  $\mathcal{I}_{\mathcal{C}}^N \subset (\pi)$ . We will show by induction on  $k$  that  $\mathcal{I}_{\mathcal{C},x} \subset (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{\mathcal{C},x}^k$ . Taking  $k = N$  we obtain 1-.

For  $k = 1$  it is obvious. Suppose that it is true for  $k - 1 \geq 1$ . It is enough to prove that  $\mathcal{I}_{\mathcal{C},x}^{k-1} \subset (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{\mathcal{C},x}^k$ . Let  $w_1, \dots, w_{k-1} \in \mathcal{I}_{\mathcal{C},x}$ . Since the image of  $\mathbf{u}_{ij}$  generates  $\mathcal{I}_{\mathcal{C},x}/(\mathcal{I}_{\mathcal{C},x}^2 + (\pi))$ , we can write  $w_p$  as

$$w_p = \lambda_p \mathbf{u}_{ij} + \pi \mu_p + \nu_p,$$

with  $\lambda_p, \mu_p \in \mathcal{O}_{\mathcal{C},x}$  and  $\nu_p \in \mathcal{I}_{\mathcal{C},x}^2$ . So we have

$$w_1 \cdots w_{k-1} = \lambda \mathbf{u}_{ij} + \pi \mu + \nu,$$

with  $\lambda, \mu \in \mathcal{O}_{\mathcal{C},x}$  and  $\nu_p \in \mathcal{I}_{\mathcal{C},x}^{2k-2}$ . Since  $2k - 2 \geq k$ , we have  $w_1 \cdots w_{k-1} \in (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{\mathcal{C},x}^k$ . This proves 1-.

We prove 2- by induction on  $p$ . The case  $p = 1$  follows 1-. Suppose that it is true for  $p - 1 \geq 1$ . So we can write  $v$  as

$$v = \lambda \mathbf{u}_{ij} + \pi^{p-1} \mu,$$

with  $\lambda, \mu \in \mathcal{O}_{C,x}$ . We can write  $v_i$  as  $v_i = \alpha\pi^p$ . So we have  $\alpha\pi_i^p = \pi_i^{p-1}\mu_i$ , whence  $\mu_i = \alpha\pi_i$ . Hence  $\mu \in \mathcal{I}_{C,x}$ . According to 1- we can write  $\mu$  as  $\mu = \theta\mathbf{u}_{ij} + \pi\tau$ , with  $\theta, \tau \in \mathcal{O}_{C,x}$ . So

$$v = (\lambda + \pi^{p-1}\theta)\mathbf{u}_{ij} + \pi^p\tau,$$

which proves the result for  $p$ .  $\square$

**4.3.6. The ideal sheaves  $\mathcal{I}_{C_I}$**  – Recall that  $I \subset \{1, \dots, n\}$  is a nonempty subset, distinct from  $\{1, \dots, n\}$ . For every subset  $J$  of  $\{1, \dots, n\}$ , let  $J^c = \{1, \dots, n\} \setminus J$  and  $\mathcal{O}_J = \mathcal{O}_{C_J}$ . It follows from proposition 4.3.3 that  $\mathcal{I}_{C_I}$  is a line bundle on  $\mathcal{C}_{I^c}$ .

From now on, we suppose that  $S \subset \mathbb{C}$  and  $P = 0$  (cf. proposition 3.2.6).

**4.3.7. Theorem:** *We have  $\mathcal{I}_{C_I} \simeq \mathcal{O}_{I^c}$ .*

*Proof.* By induction on  $n$ . If  $n = 2$  the result follows from proposition 4.2.1 and the fact that  $S \subset \mathbb{C}$ . Suppose that it is true for  $n - 1 \geq 2$ . We will prove that it is true for  $n$  by induction on the number of elements  $q$  of  $I^c$ . Suppose first that  $q = 1$  and let  $i$  be the unique element of  $I^c$ . Then according to proposition 4.3.3,  $\mathcal{I}_{C_I}$  is generated by  $(0, \dots, 0, \pi_i^{q_i}, 0, \dots, 0)$ , so the result is true in this case. Suppose that it is true if  $1 \leq q < k < n$ , and that  $q = k$ . Let  $K = \{1, \dots, n - 1\}$ . We can assume that  $I \subset K$ .

According to proposition 4.3.3, we have, for every  $x \in C$ ,  $\mathcal{I}_{C_I,x} \simeq \mathcal{O}_{I^c,x}$ . We have  $\mathcal{I}_{C_K} \subset \mathcal{I}_{C_I}$ , and  $\mathcal{I}_{C_K} \simeq \mathcal{O}_{\{n\}}$ . We have

$$\mathcal{I}_{C_I}/\mathcal{I}_{C_K} = \mathcal{I}_{C_I, C_K}$$

(the ideal sheaf of  $\mathcal{C}_I$  in  $\mathcal{C}_K$ ). From the first induction hypothesis we have

$$\mathcal{I}_{C_I, C_K} \simeq \mathcal{O}_{(I \cup \{n\})^c}.$$

So we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\{n\}} \longrightarrow \mathcal{I}_{C_I} \longrightarrow \mathcal{O}_{I^c \setminus \{n\}} \longrightarrow 0.$$

Now we will compute  $\text{Ext}_{\mathcal{O}_C}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}})$ . According to [8], 2.3, we have an exact sequence

$$0 \longrightarrow \text{Ext}_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \longrightarrow \text{Ext}_{\mathcal{O}_C}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \longrightarrow \text{Hom}(\text{Tor}_{\mathcal{O}_C}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{I^c}), \mathcal{O}_{\{n\}}).$$

Since  $\text{Tor}_{\mathcal{O}_C}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{I^c})$  is concentrated on  $\mathcal{C}_{I^c \setminus \{n\}}$ , we have

$$\text{Hom}(\text{Tor}_{\mathcal{O}_C}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{I^c}), \mathcal{O}_{\{n\}}) = \{0\}.$$

So we have

$$\text{Ext}_{\mathcal{O}_C}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) = \text{Ext}_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}).$$

Let  $\mathcal{J}$  denote the ideal sheaf of  $\mathcal{C}_{\{n\}}$  in  $\mathcal{C}_{I^c}$ . The ideal sheaf of  $\mathcal{C}_{I^c \setminus \{n\}}$  is generated by  $\mathbf{w} = (0, \dots, 0, \pi_n^m)$ , with  $m = \sum_{i \in I^c \setminus \{n\}} p_{in}$ . So we have an exact sequence of sheaves on  $\mathcal{C}_{I^c}$

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{I^c} \xrightarrow{\alpha} \mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c \setminus \{n\}} \longrightarrow 0,$$

where  $\alpha$  is the multiplication by  $\mathbf{w}$ . By the induction hypothesis there exists a surjective morphism  $\mathcal{O}_{I^c} \rightarrow \mathcal{J}$ , so we get a locally free resolution of  $\mathcal{O}_{I^c \setminus \{n\}}$

$$\mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c} \xrightarrow{\alpha} \mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c \setminus \{n\}} \longrightarrow 0 ,$$

that can be used to compute  $\mathcal{E}xt_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}})$ . It follows easily that

$$\mathcal{E}xt_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) \simeq \mathcal{O}_{\{n\}}/(\pi_n^m) .$$

We have  $\mathcal{H}om(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) = 0$ , hence

$$\begin{aligned} \text{Ext}_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}}) &\simeq H^0(\mathcal{E}xt_{\mathcal{O}_{I^c}}^1(\mathcal{O}_{I^c \setminus \{n\}}, \mathcal{O}_{\{n\}})) \\ &\simeq H^0(\mathcal{O}_{\{n\}}/(\pi_n^m)) \\ &\simeq H^0(\mathcal{O}_S/(\pi_n^m)) \\ &\simeq \mathbb{C}[\pi_n]/(\pi_n^m) . \end{aligned}$$

We will now describe the sheaves  $\mathcal{E}$  such that there exists an exact sequence

$$(2) \quad 0 \longrightarrow \mathcal{O}_{\{n\}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{I^c \setminus \{n\}} \longrightarrow 0 .$$

Let  $\nu \in \mathbb{C}[\pi_n]/(\pi_n^m)$  be associated to this exact sequence, and  $\bar{\nu} \in H^0(\mathcal{O}_S)$  over  $\nu$ . Let

$$\begin{aligned} \tau : \mathcal{O}_{\{n\}} &\longrightarrow \mathcal{O}_{\{n\}} \oplus \mathcal{O}_{I^c} \\ u &\longmapsto (\bar{\nu}u, \mathbf{w}u) \end{aligned}$$

Then according to the preceding resolution of  $\mathcal{O}_{I^c \setminus \{n\}}$  and the construction of extensions (cf. [7], 4.2), we have  $\mathcal{E} \simeq \text{coker}(\tau)$ . It is easy to see that if  $\nu = -1$  then  $\mathcal{E} \simeq \mathcal{O}_{I^c}$ . If  $\nu$  is invertible, then we have also  $\mathcal{E} \simeq \mathcal{O}_{I^c}$ , because the corresponding extension can be obtained from the one corresponding to  $\nu = -1$  by multiplying the left morphism of the exact sequence by  $\nu$ .

A similar construction can be done for extensions of  $\mathcal{O}_{I^c, x}$ -modules (for every  $x \in C$ )

$$0 \longrightarrow \mathcal{O}_{\{n\}, x} \longrightarrow V \longrightarrow \mathcal{O}_{I^c \setminus \{n\}, x} \longrightarrow 0 .$$

These extensions are classified by  $\mathcal{O}_{\{n\}, x}/(\pi_n^m)$ , and  $\mathcal{O}_{I^c, x}$  corresponds to  $-1$ .

Conversely we consider extensions

$$0 \longrightarrow \mathcal{O}_{\{n\}, x} \xrightarrow{\lambda} \mathcal{O}_{I^c, x} \xrightarrow{\mu} \mathcal{O}_{I^c \setminus \{n\}, x} \longrightarrow 0 .$$

Using the facts that  $\text{Hom}(\mathcal{O}_{\{n\}, x}, \mathcal{O}_{I^c, x})$  is generated by the multiplication by  $\mathbf{w}$  and  $\text{Hom}(\mathcal{O}_{I^c, x}, \mathcal{O}_{I^c \setminus \{n\}, x})$  by the restriction morphism, it is easy to see that  $\lambda, \mu$  are unique up to multiplication by an invertible element of  $\mathcal{O}_{I^c, x}$ . Hence the elements of  $\text{Ext}_{\mathcal{O}_{I^c, x}}^1(\mathcal{O}_{I^c \setminus \{n\}, x}, \mathcal{O}_{\{n\}, x})$  corresponding to the preceding extensions are exactly the invertible elements of  $\mathcal{O}_{\{n\}, x}/(\pi_n^m)$ .

It follows that the extensions (2) where  $\mathcal{E}$  is locally free correspond to invertible elements of  $\mathbb{C}[\pi_n]/(\pi_n^m)$ , and we have seen that in this case we have  $\mathcal{E} \simeq \mathcal{O}_{I^c}$ . Hence we have  $\mathcal{I}_{C_I} \simeq \mathcal{O}_{I^c}$  and theorem 4.3.7 is proved.  $\square$

**4.3.8. Corollary :** *The ideal sheaf of  $C_I$  is globally generated by an element  $\mathbf{u}_I$  such that for every integer  $i$  such that  $1 \leq i \leq n$  and  $i \notin I$ , the  $i$ -th coordinate of  $\mathbf{u}_I$  belongs to  $H^0(\mathcal{O}_S)$ .*

#### 4.4. PROPERTIES OF THE FRAGMENTED DEFORMATIONS

We use the notations of 4.3.

Let  $i$  be an integer such that  $1 \leq i \leq n$  and  $J_i = \{1, \dots, n\} \setminus \{i\}$ . We denote by  $\mathcal{B}$  the image of  $\mathcal{O}_C$  in  $\prod_{1 \leq j \leq n} \mathcal{O}_{C_j}/(\pi_j^{q_j})$ ; it is a sheaf of  $\mathbb{C}$ -algebras on  $C$ . Let  $\mathcal{B}_i$  be the image of  $\mathcal{O}_{C_{J_i}}$  in  $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{C_j}/(\pi_j^{q_j})$ ; it is also a sheaf of  $\mathbb{C}$ -algebras on  $C$ . For every point  $x$  of  $C$  and every  $\alpha = (\alpha_m)_{1 \leq m \leq n}$  in  $\prod_{1 \leq j \leq n} \mathcal{O}_{C_j, x}$ , we denote by  $b_i(\alpha)$  its image in  $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{C_j, x}$  (obtained by forgetting the  $i$ -th coordinate of  $\alpha$ ).

If  $p, k$  are positive integers, with  $k \leq n$ ,  $x \in C$  and  $\alpha \in \mathcal{O}_{C_k, x}$ , let  $[\alpha]_p$  denote the image of  $\alpha$  in  $\mathcal{O}_{C_k, x}/\pi_k^p$ .

**4.4.1. Proposition:** *There exists a morphism of sheaves of algebras on  $C$*

$$\Phi_i : \mathcal{B}_i \longrightarrow \mathcal{O}_{C_i}/(\pi_i^{q_i})$$

*such that for every point  $x$  of  $C$  and all  $(\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{C_{J_i}, x}$ ,  $\alpha_i \in \mathcal{O}_{C_i, x}$ , we have  $\alpha = (\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{C, x}$  if and only if  $\Phi_{i, x}(b_i(\alpha)) = [\alpha_i]_{q_i}$ .*

*Proof.* Let  $(\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{C_{J_i}, x}$ . Since  $\mathcal{C}_{J_i} \subset \mathcal{C}$ , there exists  $\alpha_i \in \mathcal{O}_{C_i, x}$  such that  $(\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{C, x}$ . If  $\alpha'_i \in \mathcal{O}_{C_i, x}$  has the same property, we have  $(0, \dots, 0, \alpha_i - \alpha'_i, 0, \dots, 0) \in \mathcal{I}_{J_i, x}$ . So according to proposition 4.3.3, we have  $[\alpha_i]_{q_i} = [\alpha'_i]_{q_i}$ . Hence we have well defined a morphism of algebras  $\theta_x : \mathcal{O}_{C_{J_i}, x} \rightarrow \mathcal{O}_{C_{J_i}}/(\pi_i^{q_i})$  sending  $(\alpha_m)_{1 \leq m \leq n, m \neq i}$  to  $[\alpha_i]_{q_i}$ . If  $j \in J_i$ , we have according to proposition 4.3.3,  $\theta_x(0, \dots, 0, \pi_j^{q_j}, 0, \dots, 0) = 0$ . Hence  $\theta_x$  induces a morphism of algebras  $\mathcal{B}_{i, x} \rightarrow \mathcal{O}_{C_i, x}/(\pi_i^{q_i})$ .  $\square$

The morphism  $\Phi_i$  has the following properties: for every point  $x$  of  $C$

- (i) For every  $\alpha = (\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{B}_{i, x}$ , we have  $\Phi_{i, x}(\alpha)|_C = \alpha_m|_C$  for  $1 \leq m \leq n$ ,  $m \neq i$ .
- (ii) We have  $\Phi_{i, x}((\pi_m)_{1 \leq m \leq n, m \neq i}) = \pi_i$ .
- (iii) Let  $j, k \in \{1, \dots, n\}$  be such that  $i, j, k$  are distinct. Let  $\mathbf{v}$  be the image of  $\mathbf{u}_{jk}$  in  $\mathcal{B}_i$ . Then there exists  $\lambda \in \mathcal{O}_{C_i, x}^*$  such that  $\Phi_{i, x}(\mathbf{v}) = \lambda \pi_i^{p_{ij}}$ .
- (iv) Let  $j$  be an integer such that  $1 \leq j \leq n$  and  $j \neq i$ . Let  $\mathbf{v}$  be the image of  $\mathbf{u}_{ij}$  in  $\mathcal{B}_{i, x}$ . Then we have  $\ker(\Phi_{i, x}) = (\mathbf{v})$ .

**4.4.2. Converse** - Let  $\mathcal{C}'$  be a glueing of  $\mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_n$  along  $C$ , which is a fragmented deformation of a primitive multiple curve of multiplicity  $n-1$ . Let  $(p_{jk})_{1 \leq j, k \leq n, j, k \neq i}$  be the spectrum of  $\mathcal{C}'$ . Let  $p_{ij}$ ,  $1 \leq j \leq n, j \neq i$  be positive integers, and  $p_{ii} = 0$ . For  $1 \leq j \leq n$ , let

$$q_j = \sum_{1 \leq k \leq n} p_{kj}.$$

Let  $\mathcal{B}_i$  be the image of  $\mathcal{O}_{\mathcal{C}'}$  in  $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{C_j}/(\pi_j^{q_j})$  and

$$\Phi_i : \mathcal{B}_i \longrightarrow \mathcal{O}_{C_i}/(\pi_i^{q_i})$$

a morphism of sheaves of algebras on  $C$  satisfying properties (i), (ii), (iii) above. Let  $\mathcal{A}$  be the subsheaf of algebras of  $\mathcal{A}$  defined by :  $\mathcal{A} = \mathcal{A}$  on  $\mathcal{A}_{top} \setminus C$ , and for every point  $x$  of  $C$ , and every  $\alpha = (\alpha_m)_{1 \leq m \leq n} \in \prod_{m=1}^n \mathcal{O}_{C_m, x}$ ,  $\alpha \in \mathcal{A}_x$  if and only if  $b_i(\alpha) \in \mathcal{B}_{i, x}$  and  $\Phi_{i, x}(b_i(\alpha)) = [\alpha_i]_{q_i}$ .

It is easy to see that  $\mathcal{A}$  is the structural sheaf of a glueing of  $C_1, \dots, C_n$  along  $C$ , which is a fragmented deformation of a primitive multiple curve of multiplicity  $n$ , and that  $\mathcal{C}' = \mathcal{A}_{\{1, \dots, i-1, i+1, \dots, n\}}$ .

We give now some applications of the preceding construction.

**4.4.3. Corollary :** *Let  $N$  an integer such that  $N \geq \max_{1 \leq i \leq n} (q_i)$ . Let  $x \in C$ ,  $\beta \in \mathcal{O}_{C_1, x} \times \dots \times \mathcal{O}_{C_n, x}$  and  $u \in \mathcal{O}_{C, x}$  such that  $u|_C \neq 0$ . Suppose that  $[\beta u]_N \in \mathcal{O}_{C, x}/(\pi^N)$ . Then we have  $[\beta]_N \in \mathcal{O}_{C, x}/(\pi^N)$ .*

*Proof.* By induction on  $n$ . It is obvious if  $n = 1$ . Suppose that the lemma is true for  $n - 1$ . Let  $I = \{1, \dots, n - 1\}$ . So we have  $[\beta|_{C_1 \times \dots \times C_{n-1}}]_N \in \mathcal{O}_{C_I, x}/(\pi_1, \dots, \pi_{n-1})^N$  by the induction hypothesis. Let  $\gamma$  (resp.  $v$ ) be the image of  $\beta$  (resp.  $u$ ) in  $\mathcal{B}_n$ . To show that  $[\beta]_N \in \mathcal{O}_{C, x}/(\pi^N)$  it is enough to verify that

$$\Phi_n(\gamma) = [\beta_n]_{q_n}.$$

We have  $\Phi_n(\gamma v) = [\beta_n u_n]_{q_n}$  because  $[\beta u]_N \in \mathcal{O}_{C, x}/(\pi^N)$ , and  $\Phi_n(v) = [u_n]_{q_n}$  because  $u \in \mathcal{O}_{C, x}$ . So we have

$$\Phi_n(\gamma)[u_n]_{q_n} = \Phi_n(\gamma)\Phi_n(v) = \Phi_n(\gamma v) = [\beta_n u_n]_{q_n} = [\beta_n]_{q_n}[u_n]_{q_n}.$$

Since  $u|_C \neq 0$ ,  $[u_n]_{q_n}$  is not a zero divisor in  $\mathcal{O}_{C_n, x}/(\pi_n^{q_n})$ , so we have  $\Phi_n(\gamma) = [\beta_n]_{q_n}$ .  $\square$

**4.4.4. Corollary :** *Let  $\mathbf{q} = \max_{1 \leq i \leq n} (q_i)$  and  $p$  the number of integers  $i$  such that  $1 \leq i \leq n$  and  $q_i = \mathbf{q}$ . Then we have  $p \geq 2$ .*

*Proof.* Suppose that  $q_i = \mathbf{q}$ . Then we have  $\pi_i^{q_i-1} \neq 0$  in  $\mathcal{O}_{C_i}/(\pi_i^{q_i})$ . Since  $\pi_i = \Phi_i((\pi_m)_{1 \leq m \leq n, m \neq i})$ , we have  $(\pi_m^{q_i-1})_{1 \leq m \leq n, m \neq i} \neq 0$  in  $\mathcal{B}_i$ . So we cannot have  $q_m < q_i$  for all the  $m \neq i$ .  $\square$

Let  $i$  be an integer such that  $1 \leq i \leq n$ ,

$$\mathcal{H} = \prod_{1 \leq j \leq n} (\pi_j^{q_j-1})/(\pi_j^{q_j}) \simeq \mathcal{O}_C^n \quad (\text{resp. } \mathcal{H}_i = \prod_{1 \leq j \leq n, j \neq i} (\pi_j^{q_j-1})/(\pi_j^{q_j}) \simeq \mathcal{O}_C^{n-1}).$$

It is an ideal sheaf of  $\prod_{1 \leq j \leq n} \mathcal{O}_{C_j}/(\pi_j^{q_j})$  (resp.  $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{C_j}/(\pi_j^{q_j})$ ). Let  $\mathcal{J} = \mathcal{H} \cap \mathcal{B}$  (resp.  $\mathcal{J}_i = \mathcal{H}_i \cap \mathcal{B}_i$ ), which is an ideal sheaf of  $\mathcal{B}$  (resp.  $\mathcal{B}_i$ ).

**4.4.5. Proposition:** *There exists a unique  $\lambda(\mathcal{C}) = (\lambda_1, \dots, \lambda_n) \in \mathbb{P}_n(\mathbb{C})$  such that for every  $\mathbf{u} = (u_j)_{1 \leq j \leq n} \in \mathcal{H}$ , we have  $\mathbf{u} \in \mathcal{J}$  if and only if  $\lambda_1 u_1 + \dots + \lambda_n u_n = 0$ . The  $\lambda_i$  are all non zero.*

*Proof.* We have  $(\pi_m)_{1 \leq m \leq n, m \neq i} \cdot \mathcal{J}_i = 0$ . Hence  $\pi_i \Phi_i(\mathcal{J}_i) = 0$  and  $\Phi_i(\mathcal{J}_i) \subset (\pi_i^{q_i-1})/(\pi_i^{q_i})$ . The restriction of  $\Phi_i, \mathcal{J}_i \rightarrow (\pi_i^{q_i-1})/(\pi_i^{q_i})$  is a morphism  $(n-1)\mathcal{O}_C \rightarrow \mathcal{O}_C$  of vector bundles on  $C$ . The existence of  $(\lambda_1, \dots, \lambda_n)$  follows from that.

If  $\lambda_i = 0$ , we have  $(0, \dots, 0, \pi_i^{q_i-1}, 0, \dots, 0) \in \mathcal{O}_C(\mathcal{C})$ . This is impossible because according to proposition 4.3.3,  $(0, \dots, 0, \pi_i^{q_i}, 0, \dots, 0)$  generates the ideal sheaf of  $\mathcal{C}_{J_i}$  in  $\mathcal{C}$ .  $\square$

For all distinct integers  $i, j$  such that  $1 \leq i, j \leq n$ , let  $I_{ij} = \{1, \dots, n\} \setminus \{i, j\}$ . Then according to proposition 4.3.3,  $\mathbf{u}_{I_{ij}i}$  generates the ideal sheaf of  $\mathcal{C}_{I_{ij}}$ . We have  $\mathbf{u}_{I_{ij}i} = (b_k)_{1 \leq k \leq n}$ , with  $b_k = 0$  if  $k \neq i, j$ ,  $b_i = \pi_i^{q_i - p_{ij}}$  and

$$b_j = \left( \prod_{1 \leq m \leq n, m \neq i, j} \alpha_{mi}^{(j)} \right) \pi_j^{q_j - p_{ij}}.$$

So we have  $\pi^{p_{ij}-1} \mathbf{u}_{I_{ij}i} \in \mathcal{J}_i$ , which gives the equation

$$(3) \quad \frac{\lambda_i}{\lambda_j} = - \prod_{1 \leq m \leq n, m \neq i, j} \mathbf{a}_{mi}^{(j)}.$$

**4.4.6. Proposition:** *For all distinct integers  $i, j, k$  such that  $1 \leq i, j, k \leq n$ , we have*

$$\mathbf{a}_{ki}^{(j)} = -\mathbf{a}_{ik}^{(j)} \mathbf{a}_{ji}^{(k)}.$$

*Proof.* We need only to treat the case  $n = 3$ , and the preceding formula by writing that  $\frac{\lambda_1}{\lambda_3} = \frac{\lambda_1}{\lambda_2} \cdot \frac{\lambda_2}{\lambda_3}$ , using (3).  $\square$

**4.4.7. Proposition:** *Let  $(\alpha_1 \pi_1^{m_1}, \dots, \alpha_n \pi_n^{m_n}) \in \mathcal{O}_{\mathcal{C}, x}$ , with  $\alpha_1, \dots, \alpha_n$  invertible. Let  $M = m_1 + \dots + m_n$ . then*

$$\left( \frac{1}{\alpha_1} \pi_1^{M-m_1}, \dots, \frac{1}{\alpha_n} \pi_n^{M-m_n} \right) \in \mathcal{O}_{\mathcal{C}, x}.$$

*Proof.* By induction on  $n$ . It is obvious for  $n = 1$ . Suppose that it is true for  $n - 1 \geq 1$ . Let  $I = \{1, \dots, n - 1\}$ . Then  $(\alpha_1 \pi_1^{m_1}, \dots, \alpha_{n-1} \pi_{n-1}^{m_{n-1}}) \in \mathcal{O}_{\mathcal{C}_I, x}$ . Hence, by the induction hypothesis, we have

$$\left( \frac{1}{\alpha_1} \pi_1^{M-m_1-m_n}, \dots, \frac{1}{\alpha_{n-1}} \pi_{n-1}^{M-m_{n-1}-m_n} \right) \in \mathcal{O}_{\mathcal{C}_I, x}.$$

So there exists  $\gamma \in \mathcal{O}_{\mathcal{C}_I, x}$  such that

$$u = \left( \frac{1}{\alpha_1} \pi_1^{M-m_1-m_n}, \dots, \frac{1}{\alpha_{n-1}} \pi_{n-1}^{M-m_{n-1}-m_n}, \gamma \right) \in \mathcal{O}_{\mathcal{C}, x}.$$

Multiplying by  $(\alpha_1 \pi_1^{m_1}, \dots, \alpha_n \pi_n^{m_n})$  we see that  $(\pi_1^{M-m_n}, \dots, \pi_{n-1}^{M-m_n}, \gamma \alpha_n \pi_n^{m_n}) \in \mathcal{O}_{\mathcal{C}, x}$ . Subtracting  $\pi^{M-m_n}$ , we find that  $(0, \dots, 0, \gamma \alpha_n \pi_n^{m_n} - \pi_n^{M-m_n}) \in \mathcal{O}_{\mathcal{C}, x}$ . There exists  $\alpha \in \mathcal{O}_{\mathcal{C}, x}$  such that the  $n$ -th coordinate of  $\alpha$  is  $\alpha_n$ , and  $\alpha$  is invertible. It follows that  $v = (0, \dots, 0, \gamma \pi_n^{m_n} - \frac{1}{\alpha_n} \pi_n^{M-m_n}) \in \mathcal{O}_{\mathcal{C}, x}$ . Now we have

$$\pi^{m_n} u - v = \left( \frac{1}{\alpha_1} \pi_1^{M-m_1}, \dots, \frac{1}{\alpha_n} \pi_n^{M-m_n} \right) \in \mathcal{O}_{\mathcal{C}, x}.$$

$\square$

**4.4.8. Corollary :** *Let  $V \subset U$  be open subsets of  $\mathcal{C}$ , and suppose that  $U \cap C \neq \emptyset$ . Let  $\alpha \in \mathcal{O}_{\mathcal{C}}(V)$  and  $\beta \in \mathcal{O}_{\mathcal{A}}(U)$  such that  $\beta|_V = \alpha$ . Then  $\beta \in \mathcal{O}_{\mathcal{C}}(U)$ .*

(Recall that  $\mathcal{A}$  is the initial glueing of  $\mathcal{C}_1, \dots, \mathcal{C}_n$  (cf. 4.1.5)).

*Proof.* This can be proved easily by induction on  $n$ , using proposition 4.4.1.  $\square$



#### 4.5. CONSTRUCTION OF FRAGMENTED DEFORMATIONS

Consider a fragmented deformation

$$\pi = \pi^{[n-1]} = (\pi_1, \dots, \pi_{n-1}) : \mathcal{C}^{[n-1]} \longrightarrow S$$

of  $C_{n-1}$ , with  $n-1$  irreducible components  $\mathcal{C}_1, \dots, \mathcal{C}_{n-1}$ . Let  $(p_{ij}^{[n-1]})_{1 \leq i, j < n}$  be its spectrum. For  $1 \leq i < n$ , let  $q_i^{[n-1]} = \sum_{1 \leq j < n} p_{ij}^{[n-1]}$ . We denote by  $\mathcal{I}_C^{[n-1]}$  the ideal sheaf of  $C$  in  $\mathcal{C}^{[n-1]}$ . Let  $\lambda(\mathcal{C}^{[n-1]}) = (\lambda_1, \dots, \lambda_{n-1})$ .

Let  $p_{1n}, \dots, p_{n-1,n}$  be positive integers,  $q_i = q_i^{[n-1]} + p_{in}$  for  $1 \leq i < n$ , and  $q_n = p_{1n} + \dots + p_{n-1,n}$ . Let  $\mathbf{u} \in \mathcal{I}_{C,x}^{[n-1]}$  whose image generates  $\mathcal{I}_{C,x}^{[n-1]} / ((\mathcal{I}_{C,x}^{[n-1]})^2 + (\pi))$ , of the form

$$\mathbf{u} = (\beta_1 \pi_1^{p_{1n}}, \dots, \beta_{n-1} \pi_{n-1}^{p_{n-1,n}}),$$

with  $\beta_i \in \mathcal{O}_{\mathcal{C}_i,x}$  invertible for  $1 \leq i < n$ .

Let  $\mathcal{B}^{[n-1]}$  be the image of  $\mathcal{O}_{\mathcal{C}^{[n-1]}}$  in  $\mathcal{O}_{\mathcal{C}_1}/(\pi_1^{q_1}) \times \dots \times \mathcal{O}_{\mathcal{C}_{n-1}}/(\pi_{n-1}^{q_{n-1}})$ . We will also denote by  $\mathbf{u}$  the image of  $\mathbf{u}$  in  $\mathcal{B}^{[n-1]}$ . Let  $\mathcal{Q} = \mathcal{B}^{[n-1]}/(\mathbf{u})$ ,  $\rho : \mathcal{B}^{[n-1]} \rightarrow \mathcal{Q}$  the projection and  $\pi_n = \rho(\pi)$ .

**4.5.1. Proposition:** *We have  $\pi_n^{q_n} = 0$ .*

*Proof.* According to proposition 4.4.7 we have

$$v = \left( \frac{1}{\beta_1} \pi_1^{q_n - p_{1n}}, \dots, \frac{1}{\beta_{n-1}} \pi_{n-1}^{q_n - p_{n-1,n}} \right) \in \mathcal{O}_{\mathcal{C}^{[n-1]},x}.$$

Hence  $\pi_n^{q_n} = v\mathbf{u} \in (\mathbf{u})$  in  $\mathcal{O}_{\mathcal{C}^{[n-1]},x}$ , and  $\pi_n^{q_n} = 0$ .  $\square$

**4.5.2. Proposition: 1** – *We have  $\pi_n^{q_n-1} = 0$  if and only if*

$$\frac{\lambda_1}{\beta_{1|C}} + \dots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} = 0.$$

*We suppose now that  $\frac{\lambda_1}{\beta_{1|C}} + \dots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \neq 0$ . Then*

**2** – *For every  $\epsilon \in \mathcal{B}_x^{[n-1]}$  such that  $\epsilon|_C \neq 0$ , we have  $\pi_n^{q_n-1}\epsilon \notin (\mathbf{u})$ .*

**3** – *For every  $\eta \in \mathcal{B}_x^{[n-1]}/(\mathbf{u})$ , and every integer  $k$  such that  $1 \leq k < q_n$ , we have  $\pi_n^k \eta = 0$  if and only if  $\eta$  is a multiple of  $\pi_n^{q_n-k}$ .*

**4** –  $\mathcal{B}_x^{[n-1]}/(\mathbf{u})$  *is a flat  $\mathbb{C}[\pi_n]/(\pi_n^{q_n})$ -module.*

*Proof.* We have  $\pi_n^{q_n-1} = 0$  if and only if  $(\pi_1^{q_n-1}, \dots, \pi_{n-1}^{q_n-1}) \in (\mathbf{u})$  in  $\mathcal{B}_x^{[n-1]}$ . We have, in  $\mathcal{O}_{\mathcal{C}_{1x}} \times \dots \times \mathcal{O}_{\mathcal{C}_{n-1,x}}$ ,

$$(\pi_1^{q_n-1}, \dots, \pi_{n-1}^{q_n-1}) = (\beta_1 \pi_1^{p_{1n}}, \dots, \beta_{n-1,n} \pi_{n-1}^{p_{n-1,n}}) \cdot \left( \frac{1}{\beta_1} \pi_1^{q_1^{[n-1]}-1}, \dots, \frac{1}{\beta_p} \pi_1^{q_{n-1}^{[n-1]}-1} \right),$$

and  $\pi_n^{q_n-1} = 0$  if and only if there exists  $\eta \in \mathcal{O}_{\mathcal{C}^{[n-1]},x}$ ,  $a_i \in \mathcal{O}_{\mathcal{C}_i,x}$ ,  $1 \leq i < n$ , such that

$$(\pi_1^{q_n-1}, \dots, \pi_{n-1}^{q_n-1}) = \eta \mathbf{u} + (a_1 \pi_1^{q_1}, \dots, a_{n-1} \pi_{n-1}^{q_{n-1}}).$$

This equality is equivalent to

$$\left(\frac{1}{\beta_1}\pi_1^{q_1^{[n-1]}-1}, \dots, \frac{1}{\beta_{n-1}}\pi_1^{q_{n-1}^{[n-1]}-1}\right) - \eta = \left(\frac{a_1}{\beta_1}\pi_1^{q_1^{[n-1]}}, \dots, \frac{a_{n-1}}{\beta_{n-1}}\pi_{n-1}^{q_{n-1}^{[n-1]}}\right).$$

Since for  $1 \leq i < n$ , we have  $(0, \dots, 0, \pi_i^{q_i^{[n-1]}}, 0, \dots, 0) \in \mathcal{O}_{\mathcal{C}^{[n-1]}, x}$ , we have  $\pi_n^{q_n-1} = 0$  if and only if

$$\left(\frac{1}{\beta_1}\pi_1^{q_1^{[n-1]}-1}, \dots, \frac{1}{\beta_{n-1}}\pi_1^{q_{n-1}^{[n-1]}-1}\right) \in \mathcal{O}_{\mathcal{C}^{[n-1]}, x}.$$

So the result of 1- follows from the définition of  $\lambda(\mathcal{C}^{[n-1]})$  (cf. prop. 4.4.5), 2- is an easy consequence.

Now we prove 3-, by induction on  $k$ . Suppose that it is true for  $k = 1$ , and that  $\pi_n^k \eta = 0$ , with  $2 \leq k < q_n$ . We have  $\pi_n^{k-1} \cdot \pi_n \eta = 0$ , so according to the induction hypothesis,  $\pi_n \eta$  is a multiple of  $\pi_n^{q_n-k+1}$ :  $\pi_n \eta = \pi_n^{q_n-k+1} \lambda$ . So  $\pi_n(\eta - \pi_n^{q_n-k} \lambda) = 0$ . Since -3 is true for  $k = 1$ , we can write  $\eta - \pi_n^{q_n-k} \lambda = \pi_n^{q_n-1} \epsilon$ , i.e.  $\eta = \pi_n^{q_n-k}(\lambda + \pi_n^{k-1} \epsilon)$ , and 3- is true for  $k$ .

It remains to prove 3- for  $k = 1$ . Suppose that  $\pi_n \eta = 0$  (with  $\eta \neq 0$ ). We can write  $\eta$  as  $\eta = \pi_n^m \theta$ , where  $\theta$  is not a multiple of  $\pi_n$ , and  $0 \leq m < q_n$ . Let  $\bar{\theta} \in \mathcal{B}_x^{[n-1]}$  be over  $\theta$ . Since  $\mathcal{I}_C = (\mathbf{u}) + (\pi)$  according to proposition 4.3.5, the condition “ $\theta$  is not a multiple of  $\pi_n$ ” is equivalent to  $\bar{\theta} \notin \mathcal{I}_{C,x}$ . We have  $\pi^{m+1} \bar{\theta} \in (\mathbf{u})$ , so according to 2-, we have  $m+1 \geq q_n$ , which proves 3- for  $k = 1$ . The last assertion is an easy consequence of 3-.  $\square$

**4.5.3. Example :** Let  $N$  be an integer,  $s \in \mathcal{O}_{\mathcal{C}^{[n-1]}, x}$  invertible, and  $k, l$  integers such that  $1 \leq k, l < n$ ,  $k \neq l$ . Suppose that for every integer  $i$  such that  $1 \leq i < n$  and  $i \neq k$  we have  $N > p_{ik}^{[n-1]}$  and  $N \geq q_i^{[n-1]} - q_k^{[n-1]} + p_{ik}^{[n-1]}$ . We take  $\mathbf{u} = \mathbf{u}_{kl} - s\pi^N$ . We have then  $\beta_i = \alpha_{kl}^{(i)}$  if  $i \neq k$ , and  $\beta_k = -s$ . The condition  $\frac{\lambda_1}{\beta_{1|C}} + \dots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \neq 0$  is fulfilled if and only if

$$\sum_{1 \leq i < n, i \neq k} \frac{\lambda_i}{\mathbf{a}_{kl}^{(i)}} - \frac{\lambda_k}{s} \neq 0.$$

**4.5.4. Construction of fragmented deformations** – Suppose that  $\frac{\lambda_1}{\beta_{1|C}} + \dots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \neq 0$ . From proposition 4.5.2, 4-, it is easy to prove that

- There exists a flat morphism of algebraic varieties  $\tau : Y \rightarrow \text{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$  with a canonical isomorphism of sheaves of  $\mathbb{C}[\pi_n]/(\pi_n^{q_n})$ -algebras  $\mathcal{O}_Y \simeq \mathcal{Q}$ , such that  $\tau^{-1}(*) = C$  (where  $*$  is the closed point of  $\text{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$ ).
- There exists a family of smooth curves  $\mathcal{C}_n$  and a flat morphism  $\pi_n : \mathcal{C}_n \rightarrow S$  extending  $\tau$  (recall that  $S$  is a germ). Hence  $Y$  is the inverse image of the subscheme of  $\mathcal{C}_n$  corresponding to the ideal sheaf  $(\pi_n^{q_n})$ . The existence of  $\mathcal{C}_n$  can be proved using Hilbert schemes of curves in projective spaces. Of course  $\mathcal{C}_n$  need not be unique.

We obtain a glueing  $\mathcal{C}$  of  $\mathcal{C}_1, \dots, \mathcal{C}_n$  by defining the sheaves of algebras  $\mathcal{O}_{\mathcal{C}}$  (on the Zariski topological space corresponding to the initial glueing  $\mathcal{A}$ ) as in 4.4.2, using for  $\Phi_n$  the quotient morphism  $\mathcal{B}^{[n-1]} \rightarrow \mathcal{Q}$ . It is easy to see that  $\pi^{-1}(P)$  is a primitive multiple curve  $C_n$  of multiplicity  $n$  extending  $C_{n-1}$ , hence  $\mathcal{C}$  is a fragmented deformation of  $C_n$ .

**4.5.5. Remark: 1** – The multiple curve  $C_n$  depends on the choice of the family  $\mathcal{C}_n$  extending the family  $Y$  parametrized by  $\text{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$ .

**2** – The multiple curve  $C_{n-1}$  is completely defined by  $\mathcal{B}^{[n-1]}$ , because  $(\pi_1^{q_1}) \times \cdots (\pi_{n-1}^{q_{n-1}}) \subset (\pi)$ . But it is not enough to know  $\mathcal{B}^{[n-1]}$  and  $\mathbf{u}$  to define  $C_n$ . In fact we need  $\mathcal{O}_{\mathcal{C}_i}/(\pi_i^{q_i+1})$ ,  $1 \leq i \leq n$ .

#### 4.6. BASIC ELEMENTS

We use the notations of 4.3 and 4.4.

Let  $\mathbf{m} = (m_1, \dots, m_n)$  be a  $n$ -tuple of positive integers, and

$$\mathbf{\Pi}^{\mathbf{m}} = (\pi_1^{m_1}) \times \cdots \times (\pi_n^{m_n}).$$

**4.6.1. Definition:** Let  $x \in C$ . An element  $u$  of  $\mathcal{O}_{\mathcal{C},x}$  is called basic at order  $\mathbf{m}$  if there exists polynomials  $P_1, \dots, P_n \in \mathbb{C}[X]$  such that

$$u \equiv (P_1(\pi_1), \dots, P_n(\pi_n)) \pmod{\mathbf{\Pi}^{\mathbf{m}}}.$$

If  $u = (P_1(\pi_1), \dots, P_n(\pi_n))$ , we say that  $u$  is basic.

Let  $\mathbf{q} = (q_1, \dots, q_n)$ . Then according to corollary 4.4.8, if  $u$  is basic at order  $\mathbf{q}$ , then for every  $y \in C$ , we have  $(P_1(\pi_1), \dots, P_n(\pi_n)) \in \mathcal{O}_{\mathcal{C},y}$ . So  $(P_1(\pi_1), \dots, P_n(\pi_n))$  is defined on a neighbourhood of  $C$ .

**4.6.2. Lemma:** Let  $u, v, w \in \mathcal{O}_{\mathcal{C},x}$  such that  $w = uv$  and  $w \neq 0$ . Suppose that  $u$  and  $w$  are basic at every order. Then  $v$  is basic at every order.

*Proof.* Let  $N$  be a positive integer such that  $N \gg 0$  and  $\mathbf{N} = (N, \dots, N)$ . Suppose that  $w \equiv (Q_1(\pi_1), \dots, Q_n(\pi_n)) \pmod{(\pi^N)}$ , where  $Q_1, \dots, Q_n \in \mathbb{C}[X]$ . Let  $\mathbf{m} = (m_1, \dots, m_n)$  be a  $n$ -tuple of positive integers, and  $v = (v_i)_{1 \leq i \leq n}$ . Suppose that

$$u \equiv (P_1(\pi_1), \dots, P_n(\pi_n)) \pmod{\mathbf{\Pi}^{\mathbf{N}}}$$

Then we have

$$Q_i(\pi_i) \equiv P_i(\pi_i).v_i \pmod{(\pi_i^N)}$$

for  $1 \leq i \leq n$ . We can write  $P_i(X)$  as  $P_i(X) = X^{n_i} R_i(X)$ , where  $R_i(X) \in \mathbb{C}[X]$  is such that  $R_i(0) \neq 0$ . Then  $Q_i(X)$  is also divisible by  $X^{n_i}$ :  $Q_i(X) = X^{n_i} S_i(X)$ , and we have in  $\mathcal{O}_{\mathcal{A}_x}$ :

$$S_i(\pi_i) \equiv R_i(\pi_i).v_i \pmod{(\pi_i^{N'})}$$

for some integer  $N' \gg 0$ . We can write  $R_i(X) = a_i.(1 - X.T_i(X))$ , with  $a_i \in \mathbb{C}^*$ ,  $T_i \in \mathbb{C}(X)$ . We have then

$$v_i \equiv \frac{S_i(\pi_i)}{a_i} \sum_{p=1}^{m_i-1} (\pi_i T_i(\pi_i))^p \pmod{\mathbf{\Pi}^{\mathbf{m}}}.$$

□

For  $1 \leq i \leq n$ , let  $\mathbf{u}_{(i)} = ((u_{(i)j})_{1 \leq j \leq n})$  be a generator of the ideal sheaf  $\mathcal{I}_{\mathcal{C}_i}$  of  $\mathcal{C}_i$  in  $\mathcal{C}$ , such that for  $1 \leq j \leq n$ ,  $u_{(i)j} \in \mathbb{C}[\pi_j]$  (cf. corollary 4.3.8).

**4.6.3. Proposition:** *Let  $v \in \mathcal{O}_{\mathcal{C},x}$ . then  $v$  is basic at every order if and only if for every  $n$ -tuple  $\mathbf{m}$  of positive integers, there exists an integer  $q > 0$  and  $P_1, \dots, P_q \in \mathbb{C}[X]$  such that*

$$v \equiv \sum_{1 \leq j \leq q} P_j(\pi) \cdot \mathbf{u}_{(i)}^j \pmod{\Pi^{\mathbf{m}}}.$$

*Proof.* We use the notations of the proof of lemma 4.6.2. Suppose that  $v = (v_j)_{1 \leq j \leq n}$  is basic at every order. Let  $N$  be a positive integer and  $\mathbf{N} = (N, \dots, N)$ . We will prove by induction on  $q \geq 0$  that we can write  $v$  as

$$(4) \quad v \equiv \sum_{0 \leq j \leq q} P_j(\pi) \cdot \mathbf{u}_{(i)}^j + \gamma_q \mathbf{u}_{(i)}^{q+1} \pmod{\Pi^{\mathbf{N}}}$$

with  $P_0, \dots, P_q \in \mathbb{C}[X]$ , and  $\gamma_q \in \mathcal{O}_{\mathcal{C},x}$ . This proves proposition 4.6.3 if  $q$  and  $N$  are big enough.

For  $q = 0$ , we have  $v_i \equiv P(\pi_i) \pmod{\pi_i^N}$ , for some  $P \in \mathbb{C}[X]$ , and we can take  $P_0 = P$ . Suppose that the result is true for  $q$  and that we have (4). Since  $v - \sum_{1 \leq j \leq q} P_j(\pi) \cdot \mathbf{u}_{(i)}^j$  is basic at any

order, using the same method as in the proof of lemma 4.6.2, we see that  $\gamma_q$  is basic at order  $\mathbf{N}'$ , where  $\mathbf{N}' = (N', \dots, N')$ , for some integer  $N' \gg 0$ . As in the case  $q = 0$  we have

$$\gamma_q \equiv P_{q+1}(\pi) + \mathbf{u}_{(i)} \cdot \gamma_{q+1} \pmod{\Pi^{\mathbf{N}'}} ,$$

with  $P_{q+1} \in \mathbb{C}[X]$ . Hence

$$v \equiv \sum_{0 \leq j \leq q+1} P_j(\pi) \cdot \mathbf{u}_{(i)}^j + \gamma_{q+1} \mathbf{u}_{(i)}^{q+2} \pmod{\Pi^{\mathbf{N}}}$$

□

**4.6.4. Proposition:** *Let  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C},x}$  be such that there exists  $P_1, \dots, P_{n-1} \in \mathbb{C}[X]$  such that, for  $1 \leq i \leq n-1$ , we have  $\alpha_i \equiv P_i(\pi_i) \pmod{(\pi_i^{q_i})}$ . Then there exists  $P_n \in \mathbb{C}[X]$  such that  $\alpha_n \equiv P_n(\pi_n) \pmod{(\pi_n^{q_n})}$ , i.e.  $\alpha$  is a basic element of order  $\mathbf{q}$ .*

*Proof.* By induction on  $n$ . The case  $n = 2$  is an easy consequence of proposition 4.2.2. Suppose that  $n \geq 3$  and that the result is true for  $n-1$ .

By subtracting multiples of  $(0, \dots, 0, \pi_i^{q_i}, 0, \dots, 0)$  we may assume that for  $1 \leq i \leq n-1$ ,  $\alpha_i \in \mathbb{C}[\pi_i]$ . By subtracting a regular function on a neighbourhood of  $C$  in  $\mathcal{C}$ , and a multiple of  $(\pi_1^{q_1}, 0, \dots, 0)$  we may also assume that  $\alpha_1 = 0$ . The ideal sheaf of  $\mathcal{C}_1$  is generated by  $\mathbf{u}_{(1)}$ . We can then write  $\alpha = \beta \mathbf{u}_{(1)}$ , with  $\beta = (\beta_i)_{1 \leq i \leq n} \in \mathcal{O}_{\mathcal{C},x}$ . We have

$$(\alpha_2, \dots, \alpha_{n-1}) = (\beta_2, \dots, \beta_{n-1}) \cdot (u_{(1)2}, \dots, u_{(1)n-1}) ,$$

hence by lemma 4.6.2,  $(\beta_2, \dots, \beta_{n-1})$  is a basic element at any order. By the induction hypothesis, there exists  $Q \in \mathbb{C}[X]$  such that  $\beta_n \equiv Q(\pi_n) \pmod{(\pi_n^{q_n - p_{1n}})}$ . Since  $u_{(1)n}$  is a multiple of  $\pi_n^{p_{1n}}$  (from the definition of  $p_{1n}$ ), it follows that  $\alpha_n \equiv u_{(1)n} Q(\pi_n) \pmod{(\pi_n^{q_n})}$ . □

#### 4.7. SIMPLE PRIMITIVE CURVES AND FRAGMENTED DEFORMATIONS

Let  $C_n$  be a primitive multiple curve of multiplicity  $n$  and associated smooth curve  $C$ . Let  $\mathcal{I}_C$  be the ideal sheaf of  $C$  in  $C_n$ . It is obvious from proposition 4.3.5, 1-, that if there exists a fragmented deformation of  $C_n$ , then we have  $\mathcal{I}_{C,C_n} \simeq \mathcal{O}_{C_{n-1}}$ , i.e.  $C_n$  is *simple* (cf. 2.4). Conversely we have

**4.7.1. Theorem:** *Let  $C_n$  be a simple primitive multiple curve of multiplicity  $n$ . Then there exists a fragmented deformation of  $C_n$ .*

*Proof.* According to theorem 2.4.1, there exists a flat family of smooth projective curves  $\tau : \mathcal{C} \rightarrow \mathbb{C}$  such that  $\tau^{-1}(0) \simeq C$  and that  $C_n$  is isomorphic to the  $n$ -th infinitesimal neighbourhood of  $C$  in  $\mathcal{C}$ . Let  $\rho_n : \mathbb{C} \rightarrow \mathbb{C}$  be the map defined by  $\rho_n(z) = z^n$ , and  $\theta = \rho_n \circ \tau : \mathcal{C} \rightarrow \mathbb{C}$ . It is a flat morphism,  $\theta^{-1}(0) = C_n$ , and for every  $z \neq 0$  in the image of  $\tau$ ,  $\theta^{-1}(z)$  is a disjoint union of  $n$  smooth irreducible curves. We can then apply the process of proposition 3.1.3 to obtain the desired fragmented deformation: it is  $\mathcal{C} \times_{\mathbb{C}} \mathbb{C}$

$$\begin{array}{ccc} \mathcal{C} \times_{\mathbb{C}} \mathbb{C} & \xrightarrow{\pi} & \mathbb{C} \\ \downarrow & & \downarrow \rho_n \\ \mathcal{C} & \xrightarrow{\theta} & \mathbb{C} \end{array}$$

□

**4.7.2. Remark:** let  $(p_{ij})$  be the spectrum of the fragmented deformation constructed in the proof of theorem 4.7.1. Then it is easy to see that  $p_{ij} = 1$  for  $1 \leq i, j \leq n$ ,  $i \neq j$ . If  $x \in C$ , then  $(\mathcal{C} \times_{\mathbb{C}} \mathbb{C})_x = \mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathbb{C},x}} \mathcal{O}_{\mathbb{C},x}$ , and if  $t = I_{\mathbb{C}} \in \mathcal{O}_{\mathbb{C},x}$ , we have for  $1 \leq k \leq n$

$$(\pi_1, \dots, \pi_{k-1}, 0, \pi_{k+1}, \dots, \pi_n) = \frac{1}{n-1} (1 \otimes t - e^{\frac{2ki\pi}{n}} (t \otimes 1)) .$$

### 5. STARS OF A CURVE

#### 5.1. DEFINITIONS

Let  $S$  be a smooth irreducible curve, and  $P \in S$  (we can also take for  $(S, P)$  a germ of smooth curve). Let  $n$  be a positive integer.

**5.1.1. Definition:** *A  $n$ -star (or more simply, a star) of  $(S, P)$  is an algebraic variety  $\mathcal{S}$  such that*

- (i)  $\mathcal{S}$  is the union of  $n$  irreducible components  $S_1, \dots, S_n$ , with fixed isomorphisms  $S_i \simeq S$ ,  $1 \leq i \leq n$ .
- (ii) For  $1 \leq i < j \leq n$ ,  $S_i \cap S_j$  has only one closed point, namely  $P$ .
- (iii) There exists a morphism  $\pi : \mathcal{S} \rightarrow S$ , which is the identity on each component  $S_i$ .

All the  $n$ -stars of  $(S, P)$  have the same underlying Zariski topological space  $S(n)$  and set of closed points. The latter is  $(\bigcup_{1 \leq i \leq n} \widehat{S}_i) / \sim$ , where  $\widehat{S}_i$  is the set of closed points of  $S_i$ , and the equivalence relation  $\sim$  is defined by: for  $x \in \widehat{S}_i$  and  $y \in \widehat{S}_j$ ,  $x \sim y$  if and only if  $i = j$  and  $x = y$ , or  $x = P \in \widehat{S}_i$  and  $y = P \in \widehat{S}_j$ . An open subset of  $\mathcal{S}$  is defined by open subsets  $U_1$  of  $S_1, \dots, U_n$  of  $S_n$ , such that for  $1 \leq i < j \leq n$ , we have  $P \in U_i$  if and only if  $P \in U_j$ .

The *initial* star  $\mathcal{S}_0$  of  $(S, P)$  is defined as follows: for every open subset  $U$  of  $S(n)$ ,  $\mathcal{O}_{\mathcal{S}_0}(U)$  is the set of  $(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{S_1}(U \cap S_1) \times \dots \times \mathcal{O}_{S_n}(U \cap S_n)$  such that if  $P \in U$  then  $\alpha_1(P) = \dots = \alpha_n(P)$ .

For every  $n$ -star  $\mathcal{S}$  of  $(S, P)$ , there is a unique dominant morphism  $\mathcal{S}_0 \rightarrow \mathcal{S}$  inducing the identity on each component. So  $\mathcal{O}_{\mathcal{S}, P}$  is a subring of  $\mathcal{O}_{\mathcal{S}_0, P}$ .

Note that (iii) is equivalent to

(iii)' For every  $\alpha \in \mathcal{O}_{S, P}$ , we have  $(\alpha, \dots, \alpha) \in \mathcal{O}_{\mathcal{S}, P}$ .

**5.1.2. Definition:** An oblate  $n$ -star (or more simply, an oblate star) of  $(S, P)$  is a  $n$ -star  $\mathcal{S}$  such that some neighbourhood of  $P$  in  $\mathcal{S}$  can be embedded in a smooth surface.

**5.1.3. Proposition:** A  $n$ -star  $\mathcal{S}$  is oblate if and only if  $\pi^{-1}(P) \simeq \text{spec}(\mathbb{C}[X]/(X^n))$ .

(cf. prop. 4.1.6).

Let  $I \subset \{1, \dots, n\}$  be a nonempty subset. Let  $\mathcal{S}^{(I)} = \bigcup_{i \in I} S_i \subset \mathcal{S}$ . If  $\mathcal{S}$  is oblate then  $\mathcal{S}^{(I)}$  is oblate too.

## 5.2. PROPERTIES OF OBLATE STARS

Let  $\mathcal{S}$  be an oblate  $n$ -star of  $S$ . Recall that  $t$  denotes a generator of the maximal ideal of  $P$  in  $S$ . We will denote this generator on  $S_i \subset \mathcal{S}$  by  $t_i$ . We will also denote by  $\pi$  the element  $t \circ \pi$  of the maximal ideal of  $P$  in  $\mathcal{S}$ . Let  $\mathcal{I}_P$  be the ideal sheaf of  $P$  in  $\mathcal{S}$ .

We begin with 2-stars:

**5.2.1. Proposition:** Suppose that  $n = 2$ . Then

- 1 – There exists a unique integer  $p > 0$  such that  $\mathcal{I}_{P, P}/(\pi)$  is generated by the image of  $(t_1^p, 0)$ .
- 2 – The image of  $(0, t_2^p)$  is also a generator of  $\mathcal{I}_{P, P}/(\pi)$ .
- 3 –  $(0, t_2^p)$  (resp.  $(t_1^p, 0)$ ) is a generator of the ideal sheaf of  $S_1$  (resp.  $S_2$ ) at  $P$ .
- 4 –  $\mathcal{O}_{S^{(2)}, P}$  consists of pairs  $(\alpha, \beta) \in \mathcal{O}_{S, P} \times \mathcal{O}_{S, P}$  such that  $\alpha - \beta \in (t^p)$ .

Now suppose that  $n \geq 2$ . Let  $I = \{i, j\} \subset \{1, \dots, n\}$ , with  $i \neq j$ . Then  $S_i \cup S_j \subset \mathcal{S}$  is a 2-star of  $S$ . Hence by proposition 5.2.1 there exists a unique integer  $p_{ij} > 0$  such that  $\mathcal{I}_{P, P}/(\pi)$  (on

$S_i \cup S_j$ ) is generated by the image of  $(t_i^{p_{ij}}, 0)$  (and also by the image of  $(0, t_j^{p_{ij}})$ ). Let  $p_{ii} = 0$ . Then the symmetric matrix  $(p_{ij})_{1 \leq i, j \leq n}$  is called the *spectrum* of  $\mathcal{S}$ .

There exists an element  $v_{ij} = (\nu_m)_{1 \leq m \leq n}$  such that  $\nu_i = 0$  and  $\nu_j = t_j^{p_{ij}}$ . For every integer  $m$  such that  $1 \leq m \leq n$ ,  $m \neq i, j$ , there exists an invertible element  $\beta_{ij}^{(m)} \in \mathcal{O}_{S,P}$  such that  $\nu_m = \beta_{ij}^{(m)} t_m^{p_{im}}$ . Let  $\beta_{ij}^{(i)} = 0$ ,  $\beta_{ij}^{(j)} = 1$ .

**5.2.2. Proposition:** *Let  $\mathbf{b}_{ij}^{(m)} = \beta_{ij}^{(m)}(P) \in \mathbb{C}$ . Then we have, for all integers  $i, j, k, m, q$  such that  $1 \leq i, j, k, m, q \leq n$ ,  $i \neq j$ ,  $i \neq k$*

$$\mathbf{b}_{ik}^{(m)} \mathbf{b}_{ij}^{(q)} = \mathbf{b}_{ik}^{(q)} \mathbf{b}_{ij}^{(m)}.$$

*In particular we have  $\mathbf{b}_{ij}^{(m)} = \mathbf{b}_{ik}^{(m)} \mathbf{b}_{ij}^{(k)}$  and  $\mathbf{b}_{ij}^{(m)} \mathbf{b}_{im}^{(j)} = 1$ .*

*For all distinct integers  $i, j, k$  such that  $1 \leq i, j, k \leq n$ , we have*

$$\mathbf{b}_{ki}^{(j)} = -\mathbf{b}_{ik}^{(j)} \mathbf{b}_{ji}^{(k)}.$$

(cf. prop. 4.3.2 and 4.4.6).

Let  $p$  an integer such that  $1 \leq p < n$ , and  $(i_1, j_1), \dots, (i_p, j_p)$   $p$  pairs of distinct integers of  $\{1, \dots, n\}$ . Then the image of  $\prod_{m=1}^p \mathbf{v}_{imjm}$  is a generator of  $(\mathcal{I}_{P,P}^p + (\pi))/(\mathcal{I}_{P,P}^{p+1} + (\pi))$ .

Let  $I \subset \{1, \dots, n\}$  be a nonempty subset, distinct from  $\{1, \dots, n\}$ . Let  $i \in \{1, \dots, n\} \setminus I$ . Let

$$\mathbf{v}_{I,i} = \prod_{j \in I} \mathbf{v}_{ji}.$$

**5.2.3. Proposition:** *The ideal sheaf of  $\mathcal{S}^{(I)}$  in  $\mathcal{S}$  is generated by  $\mathbf{v}_{I,i}$  at  $P$ .*

(cf. prop. 4.3.3).

Note that if  $I = \{1, \dots, n\} \setminus \{i\}$  then  $\mathbf{v}_{I,i|S_j} = 0$  if  $j \neq i$ , and  $\mathbf{v}_{I,i|S_i} = t_i^{q_i}$ , with  $q_i = \sum_{1 \leq j \leq n} p_{ij}$ .

Let  $i$  be an integer such that  $1 \leq i \leq n$  and  $J_i = \{1, \dots, n\} \setminus \{i\}$ . Let  $\mathcal{K}_i$  be the image of  $\mathcal{O}_{\mathcal{S}}$  in  $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{S_j}/(t_j^{q_{ji}})$ . We can view  $\mathcal{K}_i$  as a  $\mathbb{C}$ -algebra. For every  $\alpha = (\alpha_m) \in \mathcal{O}_{\mathcal{S},P}$ , let  $k_i(\alpha)$  be the image of  $\alpha$  in  $\mathcal{K}_i$ .

**5.2.4. Proposition:** *There exists a morphism of  $\mathbb{C}$ -algebras*

$$\Psi_i : \mathcal{K}_i \longrightarrow \mathcal{O}_{S_i,P}/(t_i^{q_i})$$

*such that for every  $(\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{\mathcal{S}^{(J_i)},P}$ ,  $\alpha_i \in \mathcal{O}_{S_i,P}$ , we have  $\alpha = (\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{\mathcal{S},P}$  if and only if  $\Psi_i(k_i(\alpha)) = [\alpha_i]_{q_i}$ .*

(cf. prop. 4.4.1).

The morphism  $\Psi_i$  has the following properties:

- (i) For every  $(\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{\mathcal{S}^{(j_i)}, P}$ , we have  $\Psi_i(\alpha)(P) = \alpha_m(P)$  for  $1 \leq m \leq n, m \neq i$ .
- (ii) We have  $\Psi_i((t_m)_{1 \leq m \leq n, m \neq i}) = t_i$ .
- (iii) Let  $j, k \in \{1, \dots, n\}$  be such that  $i, j, k$  are distinct. Let  $\mathbf{w}$  be the image of  $\mathbf{v}_{jk}$  in  $\mathcal{B}_i$ . Then there exists  $\lambda \in \mathcal{O}_{S_i, P}^*$  such that  $\Psi_i(\mathbf{w}) = \lambda t_i^{p_{ij}}$ .
- (iv) Let  $j$  be an integer such that  $1 \leq j \leq n$  and  $j \neq i$ . Let  $\mathbf{w}$  be the image of  $\mathbf{v}_{ij}$  in  $\mathcal{K}_i$ . Then we have  $\ker(\Psi_i) = (\mathbf{w})$ .

**5.2.5. Converse** – Let  $\mathcal{S}^{[n-1]}$  be a  $(n-1)$ -star of  $S$ , with components  $S_1, \dots, S_{n-1}$ , of spectrum  $(p_{jk})_{1 \leq j, k \leq n-1}$ . Let  $p_{nj} = p_{jn}$ ,  $1 \leq j < n$  be positive integers, and  $p_{nn} = 0$ . For  $1 \leq j \leq n$ , let  $q_j = \sum_{1 \leq k \leq n} p_{kj}$ .

Let  $S_n$  be another copy of  $S$ . Let  $\mathcal{K}_n$  be the image of  $\mathcal{O}_{\mathcal{S}^{[n-1]}}$  in  $\prod_{1 \leq j \leq n-1} \mathcal{O}_{S_j} / (t_j^{q_j})$  and

$$\Psi_n : \mathcal{K}_n \longrightarrow \mathcal{O}_{S_n} / (t_n^{q_n})$$

a morphism of  $\mathbb{C}$ -algebras satisfying properties (i), (ii), (iii) above. Let  $\mathcal{K}$  be the subsheaf of algebras of  $\mathcal{O}_{\mathcal{S}_0}$  defined by:  $\mathcal{K} = \mathcal{O}_{\mathcal{S}_0}$  on  $\mathcal{S}_0 \setminus \{P\}$ , and for every  $\alpha = (\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{\mathcal{S}_0, P}$ ,  $\alpha \in \mathcal{K}_P$  if and only if  $\Psi_n(\alpha') = [\alpha_n]_{q_n}$  (where  $\alpha'$  is the image of  $(\alpha_m)_{1 \leq m \leq n-1}$  in  $\mathcal{K}_n$ ).

It is easy to see that  $\mathcal{K}$  is the structural sheaf of an oblate  $n$ -star of  $S$ .

Let  $\mathcal{H} = \prod_{1 \leq j \leq n} (t_j^{q_j-1}) / (t_j^{q_j}) \simeq \mathbb{C}^n$  and  $\mathcal{K}$  be the image of  $\mathcal{O}_{\mathcal{S}}$  in  $\prod_{1 \leq j \leq n} \mathcal{O}_{S_j} / (t_j^{q_j})$ . We can view  $\mathcal{K}$  as a  $\mathbb{C}$ -algebra. Let  $\mathcal{J} = \mathcal{H} \cap \mathcal{K}$ .

**5.2.6. Proposition:** *There exists a unique  $\lambda(\mathcal{S}) = (\lambda_1, \dots, \lambda_n) \in \mathbb{P}_n(\mathbb{C})$  such that for every  $\mathbf{u} = (u_j)_{1 \leq j \leq n} \in \mathcal{H}$ , we have  $\mathbf{u} \in \mathcal{J}$  if and only if  $\lambda_1 u_1 + \dots + \lambda_n u_n = 0$ . The  $\lambda_i$  are all non zero.*

(cf. prop. 4.4.5).

For all distinct integers  $i, j$  such that  $1 \leq i, j \leq n$ , we have

$$\frac{\lambda_i}{\lambda_j} = - \prod_{1 \leq m \leq n, m \neq i, j} \mathbf{b}_{mi}^{(j)}.$$

### 5.3. CONSTRUCTION OF OBLATE STARS OF A CURVE

Consider an oblate  $(n-1)$ -star of  $S$ ,  $\mathcal{S}^{[n-1]}$ , with  $n-1$  irreducible components  $S_1, \dots, S_{n-1}$ , copies of  $S$ . Let  $(p_{ij}^{[n-1]})_{1 \leq i, j < n}$  be its spectrum. For  $1 \leq i < n$ , let  $q_i^{[n-1]} = \sum_{1 \leq j < n} p_{ij}^{[n-1]}$ . We

denote by  $\mathcal{I}_P^{[n-1]}$  the ideal of  $P$  in  $\mathcal{O}_{\mathcal{S}^{[n-1]}, P}$ . Let  $\lambda(\mathcal{S}^{[n-1]}) = (\lambda_1, \dots, \lambda_{n-1})$ .

Let  $p_{1n}, \dots, p_{n-1, n}$  be positive integers,  $q_i = q_i^{[n-1]} + p_{in}$  for  $1 \leq i < n$ , and  $q_n = p_{1n} + \dots + p_{n-1, n}$ . Let  $\mathbf{u} \in \mathcal{I}_P^{[n-1]}$  whose image generates  $\mathcal{I}_P^{[n-1]} / ((\mathcal{I}_P^{[n-1]})^2 + (\pi))$ , of the form

$$\mathbf{u} = (\beta_1 t_1^{p_{1n}}, \dots, \beta_{n-1} t_{n-1}^{p_{n-1, n}}),$$



with  $\beta_i \in \mathcal{O}_{S_i, P}$  invertible for  $1 \leq i < n$ .

Let  $\mathcal{K}^{[n-1]}$  be the image of  $\mathcal{O}_{\mathcal{S}^{[n-1]}}$  in  $\mathcal{O}_{S_1}/(t_1^{q_1}) \times \cdots \times \mathcal{O}_{S_{n-1}}/(t_{n-1}^{q_{n-1}})$ . We will also denote by  $\mathbf{u}$  the image of  $\mathbf{u}$  in  $\mathcal{K}^{[n-1]}$ . Let  $\mathcal{Q} = \mathcal{K}^{[n-1]}/(\mathbf{u})$ ,  $\rho : \mathcal{K}^{[n-1]} \rightarrow \mathcal{Q}$  the projection and  $t_n = \rho(\pi)$ .

**5.3.1. Proposition: 1** – We have  $t_n^{q_n} = 0$ .

**2** – We have  $t_n^{q_n-1} = 0$  if and only if

$$\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} = 0.$$

We suppose now that  $\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \neq 0$ . Then

**3** – For every  $\epsilon \in \mathcal{K}^{[n-1]}$  such that  $\epsilon(P) \neq 0$ , we have  $t_n^{q_n-1}\epsilon \notin (\mathbf{u})$ .

**4** – For every  $\eta \in \mathcal{K}^{[n-1]}/(\mathbf{u})$ , and every integer  $k$  such that  $1 \leq k < q_n$ , we have  $t_n^k \eta = 0$  if and only if  $\eta$  is a multiple of  $t_n^{q_n-k}$ .

**5** –  $\mathcal{K}^{[n-1]}/(\mathbf{u})$  is a flat  $\mathbb{C}[t_n]/(t_n^{q_n})$ -module.

**5.3.2. Construction of stars of a curve** – Suppose that  $\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \neq 0$ . From proposition 5.3.1, 5-, it is easy to prove, using 5.2.5, that there is a unique oblate  $n$ -star  $\mathcal{S}$  such that  $\mathcal{S}^{[n-1]}$  is the union  $\bigcup_{1 \leq i \leq n-1} S_i$  in  $\mathcal{S}$  and  $\Psi_n$  is the quotient map  $\mathcal{K}_n = \mathcal{K}^{[n-1]} \rightarrow \mathcal{Q}$ .

#### 5.4. MORPHISMS OF STARS

Recall that if  $\mathcal{S}$  is an oblate  $n$ -star of  $S$ , then we have a canonical inclusion of sheaves of algebras (on the underlying topological space  $S(n)$  of  $\mathcal{S}$ )  $\mathcal{O}_{\mathcal{S}} \subset \mathcal{O}_{\mathcal{S}_0}$ .

Let  $\mathcal{S}, \mathcal{S}'$  be oblate  $n$ -stars of  $S$ , with irreducible components  $S_1, \dots, S_n$ , and  $f : \mathcal{S} \rightarrow \mathcal{S}'$  a morphism inducing the identity on all the components. Such a morphism exists if and only if  $\mathcal{S}' \subset \mathcal{S}$ , and in this case  $f$  is unique and is induced by the previous inclusion. Let  $(p_{ij})$  (resp.  $(p'_{ij})$ ) be the spectrum of  $\mathcal{S}$  (resp.  $\mathcal{S}'$ ).

**5.4.1. Proposition:** We have  $p_{ij} \leq p'_{ij}$  for  $1 \leq i, j \leq n$ . If  $f$  is not the identity morphism then there exists  $i, j$  such that  $p_{ij} < p'_{ij}$ .

*Proof.* Let  $I = \{i, j\}$ . Then  $f$  induces a morphism  $\mathcal{S}^{(I)} \rightarrow \mathcal{S}'^{(I)}$ . So we have  $\mathcal{O}_{\mathcal{S}'^{(I)}, P} \subset \mathcal{O}_{\mathcal{S}^{(I)}, P}$ . From proposition 5.2.1, 4-, it follows that  $p_{ij} \leq p'_{ij}$ .

Suppose now that  $p'_{ij} = p_{ij}$  for  $1 \leq i, j \leq n$ . We must prove that  $\mathcal{S} = \mathcal{S}'$ , i.e. that  $\mathcal{O}_{\mathcal{S}'^{(I)}, P} = \mathcal{O}_{\mathcal{S}^{(I)}, P}$ . This is done by induction on  $n$ . For  $n = 2$  it is obvious. Suppose that it is true for  $n - 1$ . Let  $I = \{1, \dots, n - 1\}$ . Then  $f$  induces a morphism  $f_{n-1} : \mathcal{S}^{(I)} \rightarrow \mathcal{S}'^{(I)}$ . It follows from the induction hypothesis that  $\mathcal{S}^{(I)} = \mathcal{S}'^{(I)}$ . Since the integers  $q_i$  are the same for  $\mathcal{S}$  and  $\mathcal{S}'$ , the algebras  $\mathcal{K}_n$  for  $\mathcal{S}$  and  $\mathcal{S}'$  (cf. proposition 5.2.4) are also the same. Now let  $\alpha \in \mathcal{O}_{\mathcal{S}, P}$ , and let  $\beta \in \mathcal{K}_n$  be the image of  $\alpha$ . Let  $\alpha' \in \mathcal{O}_{\mathcal{S}', P}$  be such that its image in  $\mathcal{K}_n$  is also  $\beta$ . Then  $\alpha - \alpha'$  belongs to the ideal generated by the  $(0, \dots, 0, t_i^{q_i}, 0, \dots, 0)$ ,  $1 \leq i \leq n$ , which is included in  $\mathcal{O}_{\mathcal{S}', P}$ . Hence  $\alpha \in \mathcal{O}_{\mathcal{S}', P}$ .  $\square$

**5.4.2. Lemma:** *Suppose that  $f$  is not the identity morphism. Then there exists an ideal  $\mathcal{I} \subset \mathcal{O}_{\mathcal{S}',P}$  and  $u \in \mathcal{I}$ ,  $v \in \mathcal{O}_{\mathcal{S},P}$  such that*

$$u \otimes v \neq 0 \quad \text{in} \quad \mathcal{I} \otimes_{\mathcal{O}_{\mathcal{S}',P}} \mathcal{O}_{\mathcal{S},P}$$

and  $uv = 0$ .

*Proof.* Let  $q_1 = \sum_{i=1}^n p_{1i}$ ,  $q'_1 = \sum_{i=1}^n p'_{1i}$ . According to proposition 5.4.1 we can assume

that  $q_1 < q'_1$ . Let  $u$  be a generator of the ideal of  $S_1$  in  $\mathcal{O}_{\mathcal{S}',P}$  and  $\mathcal{I} = (u)$ . Let  $v = (t_1^{q_1}, 0, \dots, 0)$ . We have  $uv = 0$ . We have to prove that  $u \otimes v \neq 0$ . We need only to find an  $\mathcal{O}_{\mathcal{S}',P}$ -module  $M$  and a  $\mathcal{O}_{\mathcal{S}',P}$ -bilinear map

$$\phi : \mathcal{I} \otimes_{\mathcal{O}_{\mathcal{S}',P}} \mathcal{O}_{\mathcal{S},P} \longrightarrow M$$

such that  $\phi(u \otimes v) \neq 0$ . We take  $M = \mathcal{O}_{S_1,P}/(t_1^{q'_1})$ , which is a quotient of  $\mathcal{O}_{\mathcal{S}'}$ . It is easy to verify that

$$\phi : ((\lambda_i)_{1 \leq i \leq n} u, (w_i)_{1 \leq i \leq n}) \longmapsto \lambda_1 w_1 \pmod{t_1^{q'_1}}$$

is well defined, bilinear, and that  $\phi(u \otimes v) \neq 0$ .  $\square$

**5.4.3. Corollary:** *Suppose that  $f$  is not the identity morphism. Let  $Y$  be an algebraic variety and  $g : Y \rightarrow S$  a morphism such that  $g^* : \mathcal{O}_{\mathcal{S},P} \rightarrow \mathcal{O}_{Y,P}$  is injective. Then  $f \circ g : Y \rightarrow S'$  is not flat.*

*Proof.* We use the notations of the proof of lemma 5.4.2. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{S},P} & \xrightarrow{g^*} & \mathcal{O}_{Y,P} \\ \downarrow \lambda_S & & \downarrow \lambda_Y \\ \mathcal{I} \otimes_{\mathcal{O}_{\mathcal{S}',P}} \mathcal{O}_{\mathcal{S},P} & \xrightarrow{I_{\mathcal{I}} \otimes g^*} & \mathcal{I} \otimes_{\mathcal{O}_{\mathcal{S}',P}} \mathcal{O}_{Y,P} \\ \downarrow \mu_S & & \downarrow \mu_Y \\ \mathcal{O}_{\mathcal{S},P} & \xrightarrow{g^*} & \mathcal{O}_{Y,P} \end{array}$$

where  $\lambda_S(\alpha) = u \otimes \alpha$ ,  $\mu_S(u \otimes \alpha) = u\alpha$ , and  $\lambda_Y, \mu_Y$  are defined similarly. It follows that  $\mu_Y(u \otimes g^*v) = 0$ . We will show that  $u \otimes g^*v \neq 0$ , and this will imply that  $f \circ g$  is not flat. Let  $w = (t_1^{q'_1}, 0, \dots, 0)$ . Then we have  $\mathcal{I} \simeq \mathcal{O}_{\mathcal{S}',P}/(w)$ , and from the exact sequence of  $\mathcal{O}_{\mathcal{S}',P}$ -modules  $0 \rightarrow (w) \rightarrow \mathcal{O}_{\mathcal{S}',P} \rightarrow \mathcal{I} \rightarrow 0$  we deduce that  $\ker(\lambda_Y) = (w) \cdot \mathcal{O}_{Y,P}$ . Suppose that  $u \otimes g^*v = 0$ . Then  $g^*v$  is a multiple of  $w$ :  $g^*v = w \cdot a$ , for some  $a \in \mathcal{O}_{Y,P}$ . But we have  $w = g^* \pi^{q'_1 - q_1} v$ . Hence  $g^*v \cdot (1 - g^* \pi^{q'_1 - q_1}) = 0$ . Since  $1 - g^* \pi^{q'_1 - q_1}$  is invertible, we have  $g^*v = 0$ , which is false since  $g^*$  is injective. Hence  $u \otimes g^*v \neq 0$ .  $\square$

### 5.5. STRUCTURE OF IDEALS

Let  $\mathcal{S}$  be an oblate  $n$ -star of  $S$ .

**5.5.1. Proposition:** *Let  $\mathcal{I} \subset \mathcal{O}_{\mathcal{S},P}$  be a proper ideal. Then*

**1 -** *There exists a positive integer  $k$  such that  $k \leq n$  and a filtration by ideals*

$$\{0\} = \mathcal{I}_{k+1} \subset \mathcal{I}_k \subset \cdots \subset \mathcal{I}_1 = \mathcal{I}$$

*such that, for  $1 \leq i \leq k$  there exists a positive integer  $j$  such that  $j \leq n$  and an isomorphism  $\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}$  of  $\mathcal{O}_{\mathcal{S},P}$ -modules.*

**2 -** *If  $\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}$ , then  $\mathcal{I}_{i+1} \subset \mathcal{I}_{S_j}$  and  $\mathcal{I}_i \not\subset \mathcal{I}_{S_j}$ .*

*Proof.* We prove **1-** by induction on  $n$ . The case  $n = 1$  is trivial. Suppose that  $n > 1$  and that the result is true for  $n - 1$ . Let  $\mathcal{J}_1$  be the ideal sheaf of  $S_1 \subset \mathcal{S}$ , and  $\mathcal{S}' = S_2 \cup \cdots \cup S_{n-1} \subset \mathcal{S}$ . We can view  $\mathcal{J}_1$  as an ideal of  $\mathcal{O}_{\mathcal{S}',P}$ . We can suppose that  $\mathcal{I} \not\subset \mathcal{O}_{\mathcal{S}',P}$ , i.e. that some element of  $\mathcal{I}$  has a nonzero first coordinate. Let  $m$  be the smallest positive integer such that  $\mathcal{I}$  contains an element  $u$  of the form

$$u = (t^m, \alpha_2, \dots, \alpha_n) .$$

Then every element  $v$  of  $\mathcal{I}$  can be written as

$$v = \lambda u + v' ,$$

with  $\lambda \in \mathcal{O}_{\mathcal{S},P}$  and  $v' \in \mathcal{J}_1 \cap \mathcal{I}$ , and the first coordinate of  $\lambda$  is uniquely determined. It follows that  $\mathcal{I}/(\mathcal{J}_1 \cap \mathcal{I}) \simeq \mathcal{O}_{S_1,P}$ . We can apply the recurrence hypothesis to the ideal  $\mathcal{J}_1 \cap \mathcal{I}$  of  $\mathcal{O}_{\mathcal{S}',P}$  and get a filtration of it, from which we deduce the filtration of  $\mathcal{I}$ . This proves **1-** for  $n$ .

Now we prove **2-**. Let  $\alpha \in \mathcal{O}_{\mathcal{S},P} \setminus \mathcal{I}_{S_j}$ . Let  $u \in \mathcal{I}_i$  be over a generator of  $\mathcal{I}_i/\mathcal{I}_{i+1}$ . Then the image of  $\alpha u$  in  $\mathcal{I}_i/\mathcal{I}_{i+1}$  is not zero, i.e.  $\alpha u \notin \mathcal{I}_{i+1}$ . Hence  $\alpha \notin \mathcal{I}_{i+1}$ , and  $\mathcal{I}_{i+1} \subset \mathcal{I}_{S_j}$ . Let  $v_i = (0, \dots, 0, t_i^{q_i}, 0, \dots, 0) \in \mathcal{O}_{\mathcal{S},P}$ . Then the image of  $v_i u$  in  $\mathcal{I}_i/\mathcal{I}_{i+1}$  is not zero, hence  $u \notin \mathcal{I}_{S_j}$  and  $\mathcal{I}_i \not\subset \mathcal{I}_{S_j}$ .  $\square$

### 5.6. STAR ASSOCIATED TO A FRAGMENTED DEFORMATION

We keep the notations of chapter 4. Let  $n \geq 2$  be an integer,  $\pi : \mathcal{C} \rightarrow S$  a fragmented deformation of  $C_n$ , and  $\mathcal{C}_1, \dots, \mathcal{C}_n$  the irreducible components of  $\mathcal{C}$ .

Recall that  $S(n)$  is the underlying (Zariski) topological space of any  $n$ -star of  $S$ . Let  $\mathcal{C}^{top}$  be the underlying topological space of  $\mathcal{C}$ . We have an obvious continuous map  $\pi : \mathcal{C}^{top} \rightarrow S(n)$ . Let  $\mathcal{A}_n$  be the sheaf of algebras on  $S(n)$  defined by: for every open subset  $U$  of  $S(n)$ ,  $\mathcal{A}_n(U)$  is the algebra of  $(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C}}(\pi^{-1}(U))$  such that  $\alpha_i \in \mathcal{O}_{S_i}(U \cap S_i)$  for  $1 \leq i \leq n$ .

According to corollary 4.4.8, for every  $x \in C$ ,  $\mathcal{A}_{n,P}$  is the algebra of  $(\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C},x}$  such that  $\alpha_i \in \mathcal{O}_{S_i,P}$  for  $1 \leq i \leq n$ .

**5.6.1. Proposition:** *The sheaf  $\mathcal{A}_n$  is the structural sheaf of an oblate  $n$ -star of  $S$ .*

*Proof.* By induction on  $n$ . The case  $n = 1$  is obvious. Suppose that  $n > 1$  and that the result is true for  $n - 1$ . Let  $\mathcal{C}' = \mathcal{C}_1 \cup \cdots \mathcal{C}_{n-1} \subset \mathcal{C}$ , and  $\mathcal{A}_{n-1}$  the corresponding oblate  $(n - 1)$ -star of  $S$ . Let

$$\Phi_n : \mathcal{B}_n \longrightarrow \mathcal{O}_{\mathcal{C}_n} / (\pi_n^{q_n})$$

be the morphism of proposition 4.4.1. According to proposition 4.6.4,  $\Phi_n$  induces a morphism

$$\Psi_n : \mathcal{K}_n \longrightarrow \mathcal{O}_{S_n, P} / (t_n^{q_n}) .$$

By the definitions of  $\mathcal{A}_n$  and  $\Phi_n$ , if  $u = (\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{S_1, P} \times \cdots \times \mathcal{O}_{S_1, P}$ , then  $u \in \mathcal{A}_{n, P}$  if and only if  $\Psi_n(u') = v$ , where  $u'$  (resp.  $v$ ) is the image of  $u$  in  $\mathcal{K}_n$  (resp.  $\mathcal{O}_{S_n, P} / (t_n^{q_n})$ ). The result follows then from 5.2.5.  $\square$

We denote by  $\mathcal{S}(\mathcal{C})$  (or more simply  $\mathcal{S}$ ) the oblate  $n$ -star corresponding to  $\mathcal{A}_n$ , so  $\mathcal{O}_{\mathcal{S}(\mathcal{C})} = \mathcal{A}_n$ . From the definition of  $\mathcal{A}_n$  we get a canonical morphism

$$\Pi : \mathcal{C} \longrightarrow \mathcal{S}$$

such that  $\Pi|_{\mathcal{C}_i} = \pi_i : \mathcal{C}_i \rightarrow S_i$  for  $1 \leq i \leq n$ .

**5.6.2. Theorem:** *The morphism  $\Pi$  is flat.*

*Proof.* We need only to prove that  $\Pi$  is flat at any point  $x$  of  $\mathcal{C}$ . Let  $\mathcal{I} \subset \mathcal{O}_{S, P}$  be a proper ideal. We have to show that the canonical morphism of  $\mathcal{O}_{S, P}$ -modules

$$\tau = \tau_{\mathcal{I}} : \mathcal{O}_{\mathcal{C}, x} \otimes_{\mathcal{O}_{S, P}} \mathcal{I} \longrightarrow \mathcal{O}_{\mathcal{C}, x}$$

is injective. According to proposition 5.5.1 there is a filtration by ideals

$$\{0\} = \mathcal{I}_{k+1} \subset \mathcal{I}_k \subset \cdots \subset \mathcal{I}_1 = \mathcal{I}$$

such that, for  $1 \leq i \leq k$  there exists a positive integer  $j$  such that  $j \leq n$  and an isomorphism  $\mathcal{I}_i / \mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j, P}$  of  $\mathcal{O}_{S, P}$ -modules. We will prove the injectivity of  $\tau$  by induction on  $k$ .

Recall that for  $1 \leq j \leq n$ ,  $\mathcal{I}_{S_j, P} = \mathcal{I}_{S_j, S, P}$  is a principal ideal, generated by an element  $u_j$  which is also a generator of  $\mathcal{I}_{\mathcal{C}_j, x} = \mathcal{I}_{\mathcal{C}_j, \mathcal{C}, x}$  (cf. corollary 4.3.8 and proposition 5.2.3), and that the only zero coordinate of  $u_j$  is the  $j$ -th.

Suppose that  $k = 1$ , so  $\mathcal{I}$  is isomorphic to  $\mathcal{O}_{S_j, P}$  for some  $j$ . Let  $u$  be a generator of  $\mathcal{I}$  and  $w \in \mathcal{O}_{\mathcal{C}, x} \otimes_{\mathcal{O}_{S, P}} \mathcal{I}$ , that can be written as  $w = v \otimes u$ ,  $v \in \mathcal{O}_{\mathcal{C}, x}$ . Suppose that  $\tau(v \otimes u) = vu = 0$ . Since  $\mathcal{I}$  is annihilated by  $\mathcal{I}_{S_j, P}$ , we have  $\mathcal{I} \subset ((0, \dots, 0, t_j^{q_j}, 0, \dots, 0))$ . Since  $vu = 0$ , the  $j$ -th component of  $v$  is zero, i.e.  $v \in \mathcal{I}_{\mathcal{C}_j, x}$ . Hence  $v$  is a multiple of  $u_j$  :  $v = \alpha u_j$ . We have then

$$\begin{aligned} w &= \alpha u_j \otimes u \\ &= \alpha \otimes u_j u \quad (\text{because } u_j \in \mathcal{O}_{S, P}) \\ &= 0 \quad (\text{because } u_j u = 0) . \end{aligned}$$

Hence  $\tau$  is injective.

Suppose that the result is true for  $k - 1 \geq 1$  and that the filtration of  $\mathcal{I}$  is of length  $k$ . According to proposition 5.5.1, **1-**, we have  $\mathcal{I} / \mathcal{I}_2 \simeq \mathcal{O}_{S_j, P}$  for some  $j$ . Let  $u \in \mathcal{I}$  be such that its image in  $\mathcal{I} / \mathcal{I}_2$  is a generator, and  $w \in \mathcal{O}_{\mathcal{C}, x} \otimes_{\mathcal{O}_{S, P}} \mathcal{I}$  such that  $\tau(w) = 0$ . We can write  $w$  as  $w = \alpha \otimes v + \beta \otimes u$ , with  $\alpha, \beta \in \mathcal{O}_{\mathcal{C}, x}$  and  $v \in \mathcal{I}_2$ . Since  $\alpha v + \beta u = 0$ , we have  $\beta u \in \mathcal{O}_{\mathcal{C}, x} \mathcal{I}_2$ ,

and  $\mathcal{O}_{\mathcal{C},x}\mathcal{I}_2 \subset \mathcal{I}_{\mathcal{C}_j}$  by proposition 5.5.1, **2-**, i.e. the  $j$ -th coordinate of  $\beta u$  is zero. By proposition 5.5.1, **2-**, the  $j$ -th coordinate of  $u$  does not vanish, hence the  $j$ -th coordinate of  $\beta$  is zero, i.e.  $\beta \in \mathcal{I}_{\mathcal{C}_j}$ . Hence  $\beta$  is a multiple of  $u_j$  :  $\beta = \gamma u_j$ . We have then

$$\beta \otimes u = \gamma u_j \otimes u = \gamma \otimes u_j u,$$

and  $u_j u \in \mathcal{I}_2$  (because its image in  $\mathcal{I}/\mathcal{I}_2$  vanishes). It follows that  $w$  is the image of an element  $w'$  of  $\mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{S,P}} \mathcal{I}_2$ . We have  $\tau_{\mathcal{I}_2}(w') = 0$ , hence by the induction hypothesis  $w' = 0$ . It follows that we have also  $w = 0$ .  $\square$

**5.6.3. Remark:** If  $\mathcal{S}'$  is an oblate  $n$ -star of  $S$ , and if  $\Pi' : \mathcal{C} \rightarrow \mathcal{S}'$  is a flat morphism compatible with the projections to  $S$ , then we have  $\mathcal{S}' = \mathcal{S}(\mathcal{C})$  and  $\Pi' = \Pi$ . This is an easy consequence of corollary 5.4.3.

**5.6.4. Converse** - Let  $\pi : \mathcal{S} \rightarrow S$  be an oblate  $n$ -star of  $S$ . Let  $\Pi : \mathcal{C} \rightarrow \mathcal{S}$  be a flat morphism such that for every closed point  $s \in \mathcal{S}$ ,  $\Pi^{-1}(s)$  is a smooth irreducible projective curve. Let  $C = \Pi^{-1}(P)$  and  $\tau = \pi \circ \Pi : \mathcal{C} \rightarrow S$ . Then  $C_n = \tau^{-1}(P)$  is a primitive multiple curve of multiplicity  $n$  and associated smooth curve  $C$ , and  $\mathcal{C}$  is a fragmented deformation of  $C_n$ . This is an easy consequence of proposition 4.1.6.

## 6. CLASSIFICATION OF FRAGMENTED DEFORMATIONS OF LENGTH 2

Let  $\pi : \mathcal{C} \rightarrow \mathbb{C}$  be a fragmented deformation of length 2. The corresponding double curve  $C_2$  is  $\pi^{-1}(0)$ . Suppose that the spectrum of  $\mathcal{C}$  is  $\begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$ . This means that the infinitesimal neighbourhoods of order  $p$  of  $C$  in  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are isomorphic, i.e. we have an isomorphism of sheaves of algebras on  $C$

$$\Phi : \mathcal{O}_{\mathcal{C}_1}/(\pi_1^p) \longrightarrow \mathcal{O}_{\mathcal{C}_2}/(\pi_2^p),$$

and for every point  $x$  of  $C$ , we have

$$\mathcal{O}_{\mathcal{C},x} = \{(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} ; \alpha_2 \pmod{\pi_2^p} = \Phi(\alpha_1 \pmod{\pi_1^p})\}.$$

Let  $C_i^k$  denote the infinitesimal neighbourhood of order  $k$  of  $C$  in  $\mathcal{C}_i$ ,  $i = 1, 2$ ,  $k > 0$ . It is a primitive multiple curve of multiplicity  $k$  and associated smooth curve  $C$ , and we have  $C_1^p = C_2^p$ . Hence  $C_1^{p+1}$  and  $C_2^{p+1}$  appear as extensions of  $C_1^p$  in primitive multiple curves of multiplicity  $p+1$ . According to [6] and [10] these extensions are classified by  $H^1(C, T_C)$  ( $T_C$  being the tangent sheaf on  $C$ ). More precisely, we say that two such extensions  $D$ ,  $D'$  are *isomorphic* if there exists an isomorphism  $D \simeq D'$  leaving  $C_1^p$  invariant. Then if  $\mathcal{H}$  is the set of isomorphism classes of such extensions, a bijection  $\lambda : H^1(C, T_C) \rightarrow \mathcal{H}$  is defined in [6], such that  $\lambda(0) = C_1^{p+1}$ .

On the other hand, it follows from [2], [6] that the primitive double curves with associated smooth curve  $C$  and associated line bundle  $\mathcal{O}_C$  are classified by  $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ .

**6.0.5. Theorem:** *The point of  $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$  corresponding to  $C_2$  is  $\mathbb{C} \cdot \lambda^{-1}(C_2^{p+1})$ .*

*Proof.* According to [6], there exists an open covering  $(U_i)_{i \in I}$  of  $C$  such that for  $k = 1, 2$ , the open subset of  $C_k^{p+1}$  corresponding to  $U_i$  is isomorphic to  $U_i \times \text{spec}(C[t]/(t^{p+1}))$ . Here  $t$  is  $\pi_1$  on  $\mathcal{C}_1$  and  $\pi_2$  on  $\mathcal{C}_2$ . We obtain then cocycles  $(\theta_{ij}^{(k)})_{i,j \in I}$ , where  $\theta_{ij}^{(k)}$  is an automorphism of  $U_{ij} \times \text{spec}(C[t]/(t^{p+1}))$ . We can also suppose that  $\omega_{C|U_i}$  is trivial, for every  $i \in I$ . Let  $dx_{ij} = dx$  be a generator of  $\omega_C(U_{ij})$ . Since the ideal sheaf of  $C$  in  $C_k^{p+1}$  is the trivial sheaf on  $C_k^p$ , we can write, using the notations of [6],  $\theta_{ij}^{(k)} = \phi_{\mu_{ij}^{(k)}, 1}$ , with  $\mu_{ij}^{(k)} \in \mathcal{O}_C(U_{ij})[t]/(t^p)$ , i.e. for every  $\alpha \in \mathcal{O}_C(U_i)$ , we have, at the level of regular functions

$$\theta_{ij}^{(k)}(\alpha) = \sum_{m=0}^p \frac{1}{m!} (\mu_{ij}^{(k)} t)^m \frac{d^m \alpha}{dx^m},$$

and  $\theta_{ij}^{(k)}(t) = t$ . Since  $C_1^p = C_2^p$  we can suppose that  $\mu_{ij}^{(1)} \equiv \mu_{ij}^{(2)} \pmod{t^{p-1}}$ . Hence  $\tau_{ij} = \mu_{ij}^{(2)} - \mu_{ij}^{(1)} \in (t^{p-1})/(t^p) \simeq \mathcal{O}_C(U_i)$ . The family  $(\tau_{ij})$  is (in some sense) a cocycle representing  $\lambda^{-1}(C_2^{p+1})$  (cf. [6], [10]).

We have  $(\pi_1^{p+1}) + (\pi_2^{p+1}) \subset (\pi)$  in  $\mathcal{O}_C$ . Hence  $C_2 = \pi^{-1}(0)$  is contained in the subscheme  $Z$  of  $\mathcal{C}$  corresponding to the ideal sheaf  $(\pi_1^{p+1}) + (\pi_2^{p+1})$ . We have

$$\begin{aligned} \mathcal{O}_Z(U_{ij}) &= \{(\alpha_1, \alpha_2) \in \mathcal{O}_{C_1}(U_{ij})/(t^{p+1}) \times \mathcal{O}_{C_2}(U_{ij})/(t^{p+1}) ; \Phi(\alpha_1 \bmod t^p) = \alpha_2 \bmod t^p\} \\ &= \{(\alpha_1, \alpha_2) \in \mathcal{O}_C(U_{ij})[t]/(t^{p+1}) \times \mathcal{O}_C(U_{ij})[t]/(t^{p+1}) ; \alpha_1 \equiv \alpha_2 \bmod t^p\}. \end{aligned}$$

To obtain  $\mathcal{O}_{C_2}(U_{ij})$ , we have just to quotient by  $\pi = (t, t)$ , and we obtain

$$\mathcal{O}_{C_2}(U_{ij}) = \mathcal{O}_Z(U_{ij})/(t, t) \simeq \mathcal{O}_C(U_{ij})[z]/(z^2),$$

the last isomorphism beeing

$$(a_0 + a_1 t + \cdots + a_{p-1} t^{p-1} + \alpha t^p, a_0 + a_1 t + \cdots + a_{p-1} t^{p-1} + \beta t^p) \mapsto \alpha_0 + (\beta - \alpha)z.$$

Now we can explicit the automorphism of  $\mathcal{O}_C(U_{ij})[z]/(z^2)$  induced by  $\theta_{ij}$  (these isomorphisms will define the cocycle corresponding to  $C_2$ ). It is easy to see that this isomorphism is  $\phi_{\tau_{ij}, 1}$ , which proves theorem 6.0.5.  $\square$

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